Analytic continuation over complex landscapes

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Introduction

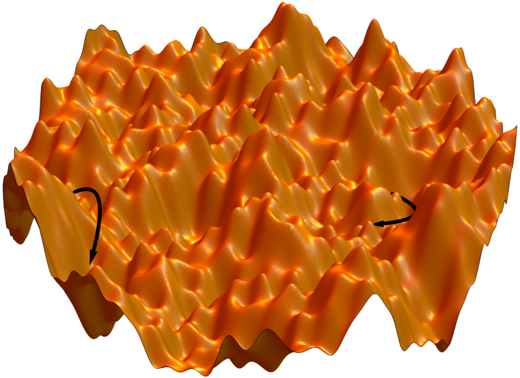
Why analytic continuation?

- Formally define a formally undefined theory...
- Calculate something in an oscillatory theory...

...by continuing from a well-defined and nonoscillatory region.

Tiny outline

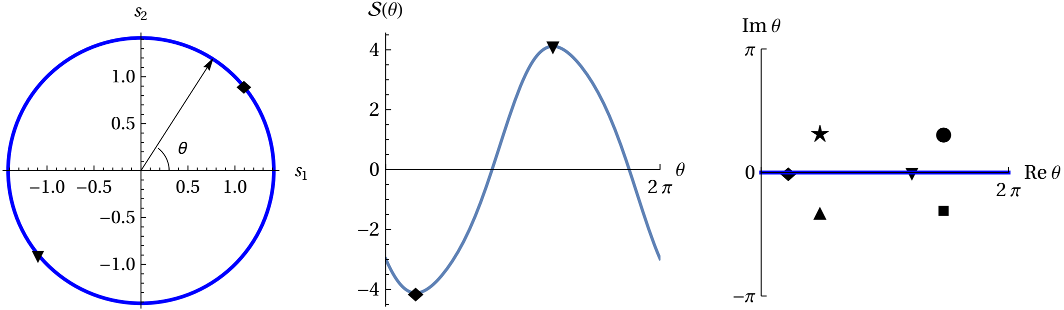
1. How to continue phase space integrals
2. The structure of complex phase space in 'complex' theories
3. Implications for continuation



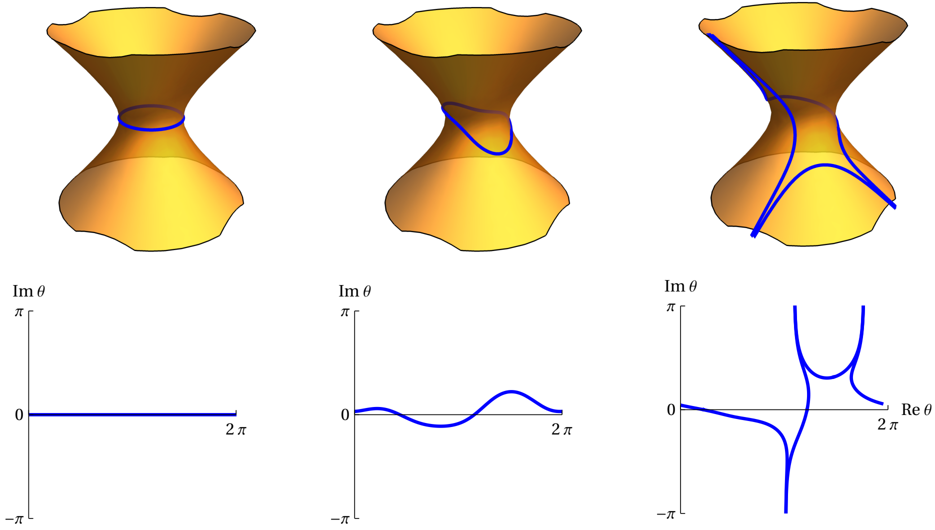
Chiara Cammarota via Simons Glass Collaboration

3-spin spherical model: N-component 'spin' s with

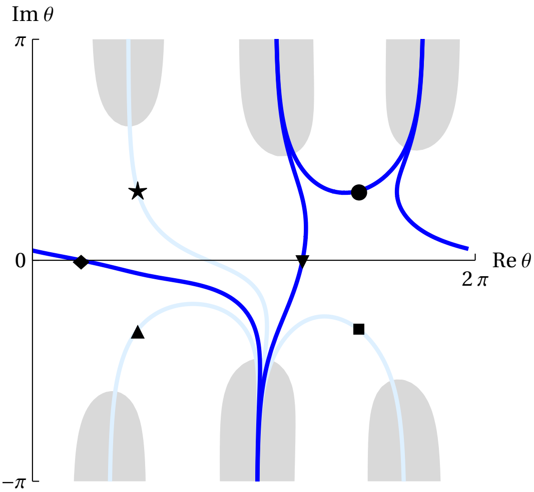
$$\mathcal{S}(s) = \frac{1}{3!}\sum_{i,j,k}^{N} J_{ijk}\, s_i s_j s_k \qquad N = s^2 = (\mathrm{Re}\, s)^2 - (\mathrm{Im}\, s)^2$$



"Circular model" with $N = 2$, parameterized by $\theta = \arctan(s_2/s_1)$



$$Z = \int_{S^{N-1}} ds\, e^{-\beta \mathcal{S}(s)} = \oint_{\mathcal{C}} ds\, e^{-\beta \mathcal{S}(s)} = \sum_{\sigma} n_{\sigma} \oint_{\mathcal{J}_{\sigma}} ds\, e^{-\beta \mathcal{S}(s)}$$

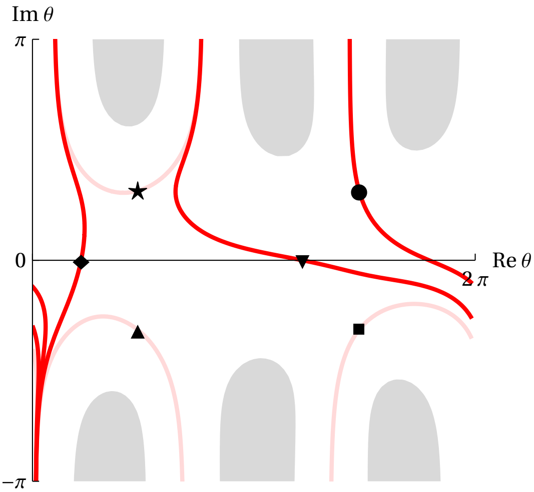


$$Z = \sum_{\sigma} n_{\sigma} \oint_{\mathcal{J}_{\sigma}} ds\, e^{-\beta \mathcal{S}(s)}$$

$$\mathcal{C} = \mathcal{J}_{\blacklozenge} + \mathcal{J}_{\blacktriangledown} + \mathcal{J}_{\bullet}$$

Thimble $\mathcal{J}_{\sigma}$: set of all points that approach the stationary point σ under gradient descent on Re $\beta\mathcal{S}$

- are surfaces of constant phase
- connect good regions where the integrand vanishes
- form a basis for valid contours



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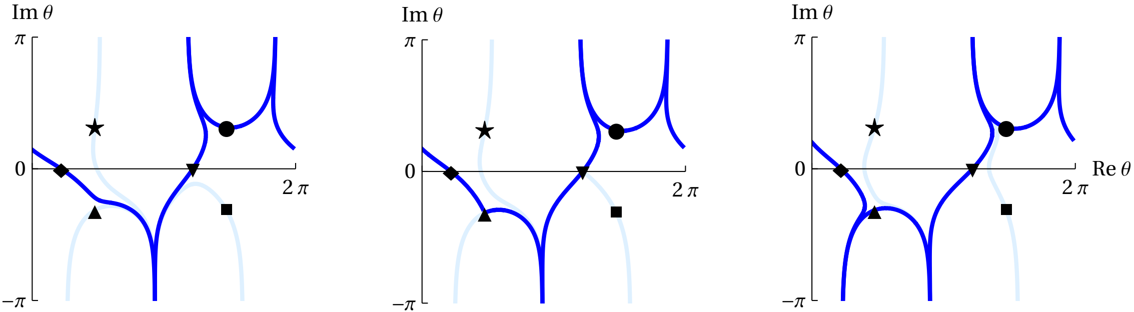
Smooth continuation of parameters like β mostly doesn't change thimble decomposition, but sometimes does at *Stokes points*.

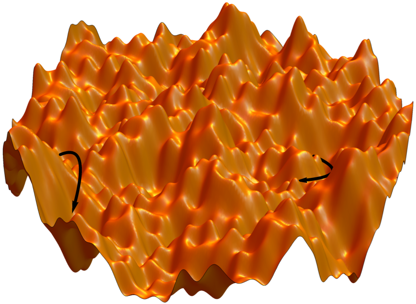


$\mathcal{C} = \mathcal{J}_{\blacklozenge} + \mathcal{J}_{\blacktriangledown} + \mathcal{J}_{\bullet}$

$\mathcal{C} = ???$

$\mathcal{C} = \mathcal{J}_{\blacklozenge} + \mathcal{J}_{\blacktriangle} + \mathcal{J}_{\blacktriangledown} + \mathcal{J}_{\bullet}$

Complex landscapes have a superextensive number of stationary points: $\mathcal{N} \sim e^{N\Sigma}$



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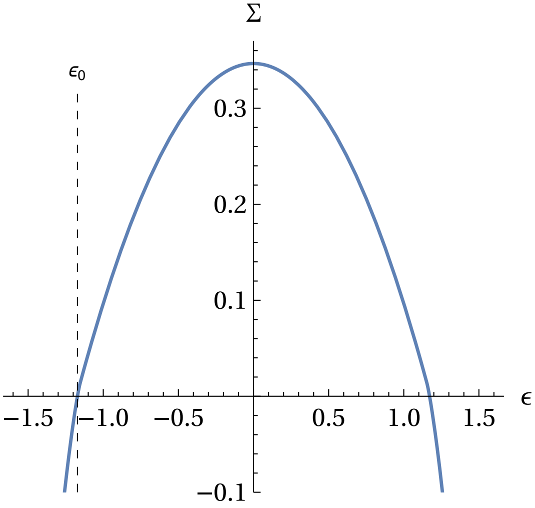
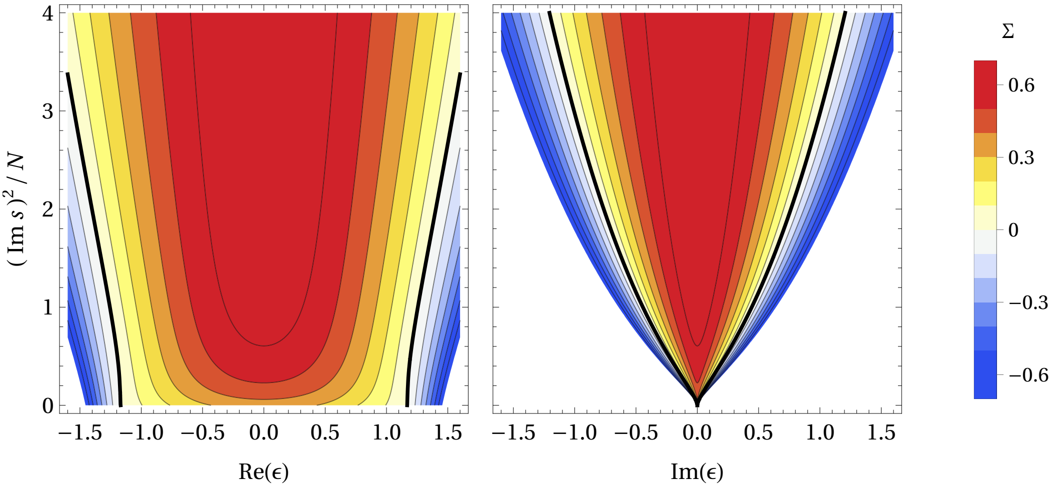
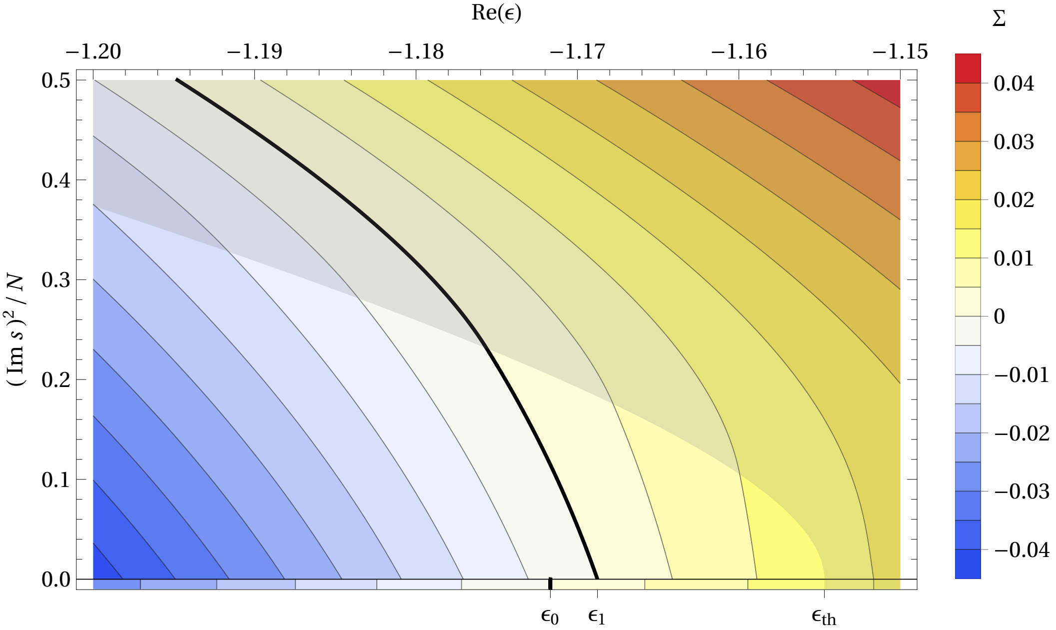


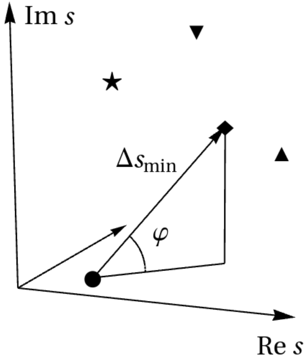


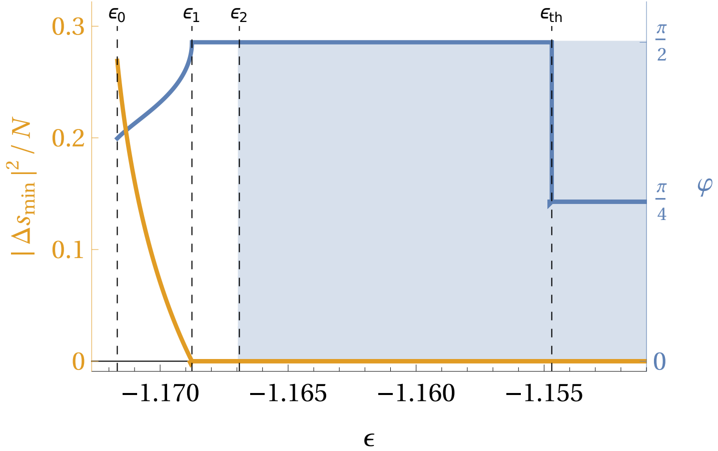
(Im s)^2 / N
Re(ϵ)
Im(ϵ)
Σ
0.6
0.3
0
−0.3
−0.6
4
3
2
1
0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5



Re(ϵ)
−1.20
−1.19
−1.18
−1.17
−1.16
−1.15
Σ
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
0.5
0.4
0.3
0.2
0.1
0.0
(Im s)² / N
ϵ₀
ϵ₁
ϵth

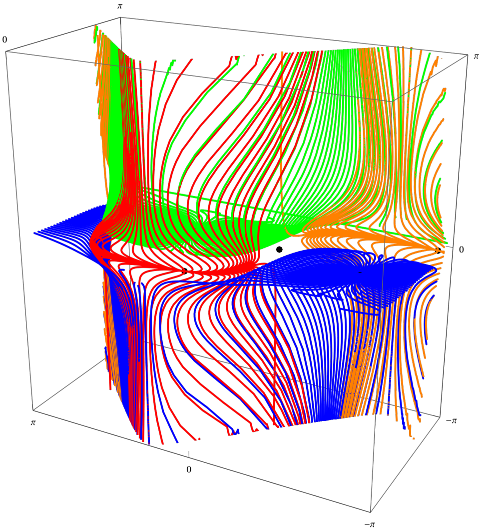
Relative position of nearest-neighbor stationary points in the landscape shapes the propensity for Stokes points





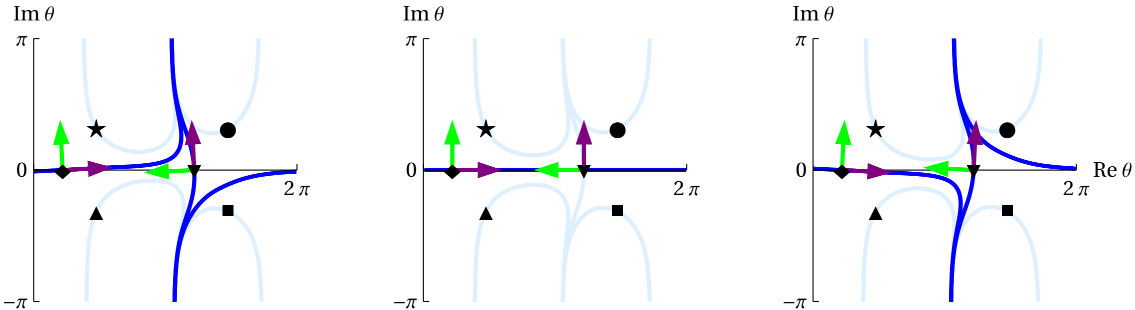
Summary

- Analytic continuation of exponential integrals relies on decomposition into thimbles attached to stationary points
- 'Complex' landscapes with many stationary points have even more in their complex extensions
- Relative position of nearby stationary points in complex phase space changes dramatically at the 'threshold'



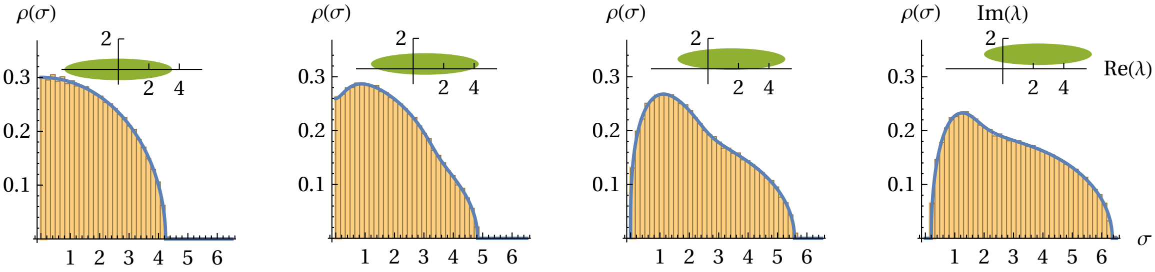
Complex complex landscapes, JK-D & J Kurchan, PRR **3** 023064 (2021) & forthcoming. . .

Bonus: Thimble orientation and the weights n_σ



$\mathcal{C} = \mathcal{J}_\blacklozenge - \mathcal{J}_\blacktriangledown$ $\quad$ $\mathcal{C} = \mathcal{J}_\blacklozenge$ $\quad$ $\mathcal{C} = \mathcal{J}_\blacklozenge + \mathcal{J}_\blacktriangledown$

Bonus: Threshold energy in the complex phase space



$|\epsilon| = 0$ $\quad$ $|\epsilon| < |\epsilon_{\text{gap}}|$ $\quad$ $|\epsilon| = |\epsilon_{\text{gap}}|$ $\quad$ $|\epsilon| > |\epsilon_{\text{gap}}|$