How to count in hierarchical landscapes

complexity in the mixed spherical models

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INFN Sezione di Roma I

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height
(−E)


height
(−ξ)


height
(−ε)


fewer, clustered
more, evenly dispersed
height
(−E)


very stable


very stable
less stable


very stable
marginally stable
less stable


very stable
unstable (saddle)
marginally stable
less stable

What landscapes?

Gaussian random functions H on the N-dimensional hypersphere with

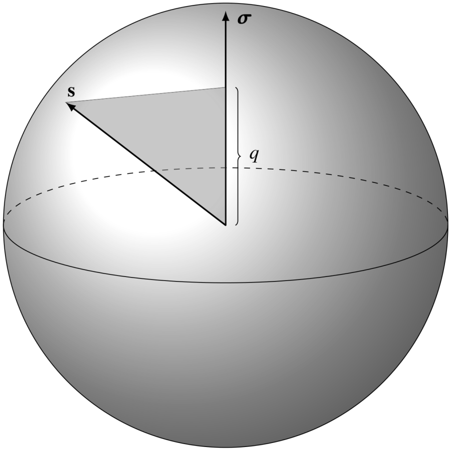
$$\overline{H(\boldsymbol{s})H(\boldsymbol{\sigma})} = \frac{1}{N} f\left(\frac{\boldsymbol{s}\cdot\boldsymbol{\sigma}}{N}\right)$$

Important to

- Mean-field glass and spin-glass physics (*pure and mixed spherical models*)
- Inference applied to signal detection (*spiked tensor model*)

This talk: mixed $p + s$ models of the form

$$f(q) = \frac{1}{2}[\lambda q^p + (1-\lambda)q^s]$$



How to count stationary points

Number of points given by integral

$$\mathcal{N}(E,\mu) = \int d\nu(s \mid E,\mu) \sim e^{N\Sigma(E,\mu)}$$

over Kac–Rice measure

$$d\nu(s \mid E,\mu) = \overbrace{\delta\big(\nabla H(s)\big)\,\big|\det \operatorname{Hess} H(s)\big|}^{\text{All stationary points...}}\ \underbrace{\delta\big(H(s)-NE\big)}_{\text{with energy density } E}\ \underbrace{\delta\big(\operatorname{Tr}\operatorname{Hess} H(s)-N\mu\big)}_{\text{and stability } \mu.}$$

Two ways to average the count to measure complexity:

$$\underbrace{\Sigma = \frac{1}{N}\overline{\log \mathcal{N}(E,\mu)}}_{\text{'quenched' average}} \quad \leq \quad \underbrace{\Sigma_a = \frac{1}{N}\log \overline{\mathcal{N}(E,\mu)}}_{\text{'annealed' average}}$$

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Characterizing stability

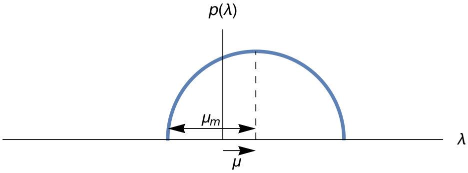
Type of point controlled by the *stability*

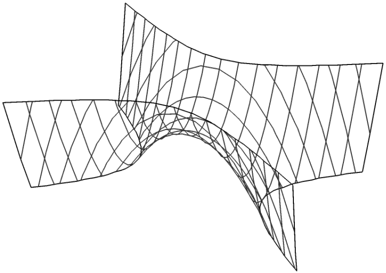
$$\mu = \frac{1}{N} \operatorname{Tr} \operatorname{Hess} H(s^*)$$

For $\mu < \mu_\mathrm{m}$, stationary points are saddles with varying index

For $\mu > \mu_\mathrm{m}$, stationary points are minima with varying stiffness

For $\mu = \mu_\mathrm{m}$, stationary points are marginal minima





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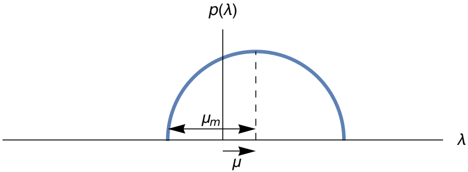

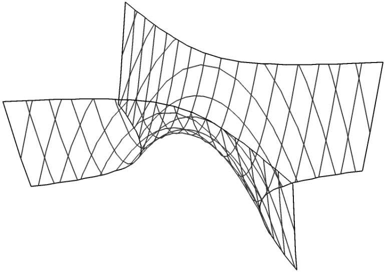

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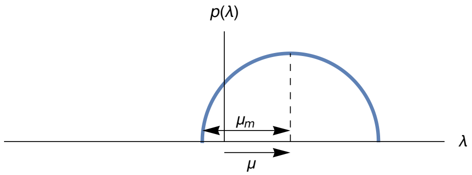

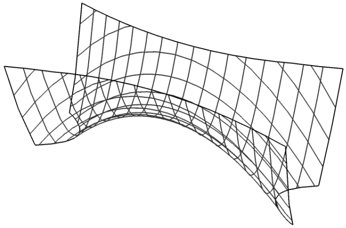

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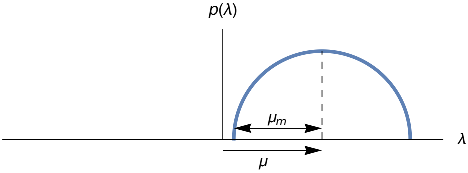

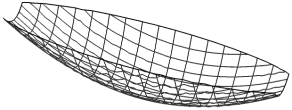

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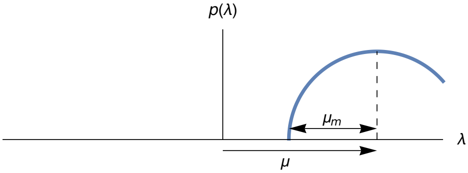

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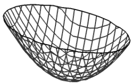

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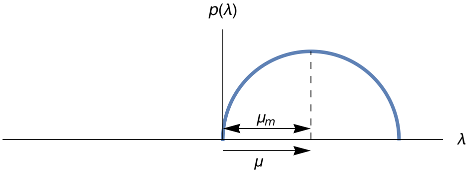
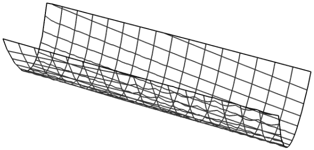




How to count stationary points

After manipulations, (quenched) complexity given by integral over three $n \times n$ matrix order parameters (and two scalar order parameters)

$$\Sigma(E,\mu) = \frac{1}{N} \lim_{n\to 0} \frac{\partial}{\partial n} \int dC\, dR\, dD\, d\hat\beta\, d\hat\mu\, e^{nN\mathcal{S}(C,R,D,\hat\beta,\hat\mu)}$$

C, R, D describe clustering structure of stationary points with $(E,\ \mu)$

$$\mathcal{S}(C,R,D,\hat\beta,\hat\mu) = \mathcal{D}(\mu) + \hat\beta E - \frac{1}{2}\hat\mu + \frac{1}{n}\Bigg(\frac{1}{2}\hat\mu \operatorname{Tr} C - \mu \operatorname{Tr} R$$
$$+\frac{1}{2}\sum_{ab}\left[\hat\beta^2 f(C_{ab}) + (2\hat\beta R_{ab} - D_{ab})f'(C_{ab}) + R_{ab}^2 f''(C_{ab})\right] + \frac{1}{2}\ln\det\begin{bmatrix} C & iR \\ iR & D \end{bmatrix}\Bigg)$$

Evaluating integral by method of steepest descent requires finding saddles of $\mathcal{S}$

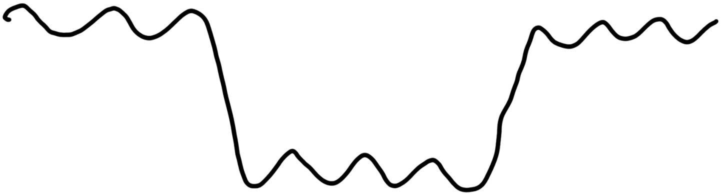

HIGH PEAKS

How to count near the ground state

There is an exact correspondence between zero-temperature limit of equilibrium and ground state of complexity:

$$\lim_{T\to 0} Q \quad \Longleftrightarrow \quad \lim_{E\to E_{\mathbf{gs}}} [C, D, R, \hat{\beta}, \hat{\mu}]$$

If Q is kRSB then C, D, R are $(k-1)$RSB



Strategy: analytic continuation of ground state order parameters to the entire phase

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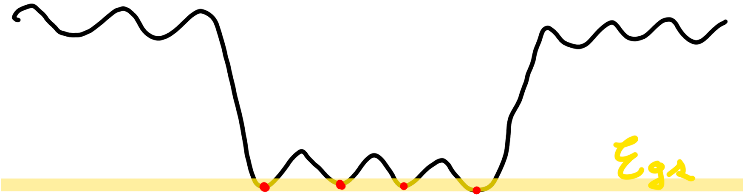

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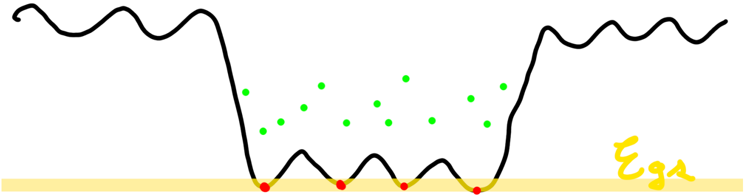

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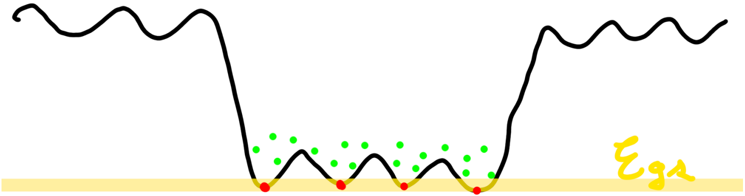

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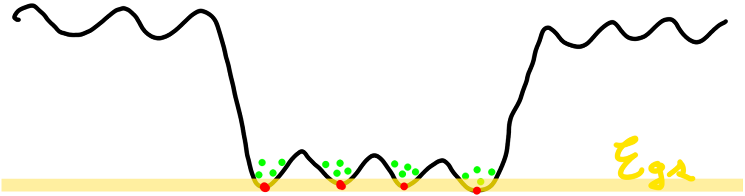

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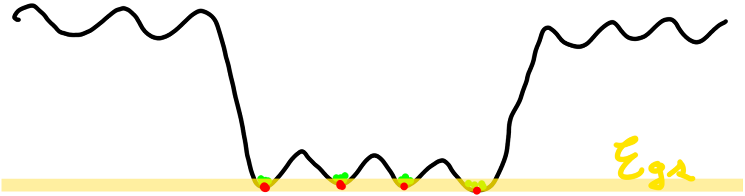

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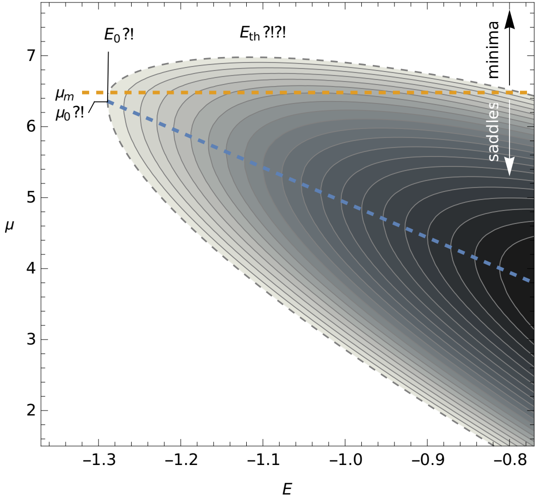
How to count in hierarchical landscapes: 1RSB

Example: $3 + 16$ model with 2RSB equilibrium. Annealed complexity predicts *saddles* at ground state, not minima.

Quenched complexity predicts 1RSB clustering among certain minima and saddles, consistent ground state

Spherical $3 + 16$ model: $f(q) = \frac{1}{2}(q^3 + \frac{1}{16}q^{16})$



A Crisanti and L Leuzzi, "Amorphous-amorphous transition and the two-step replica symmetry breaking phase", Physical Review B **76**, 184417 (2007)

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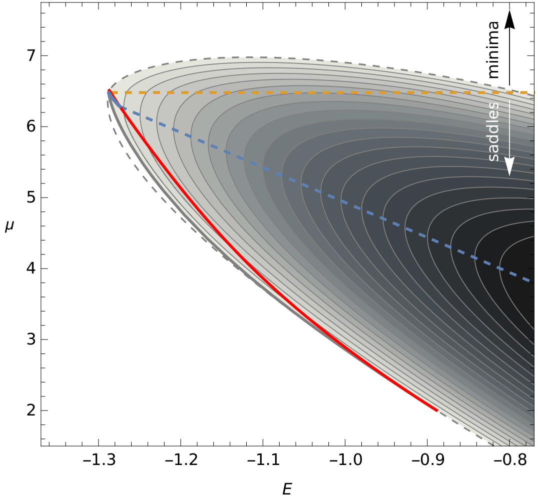

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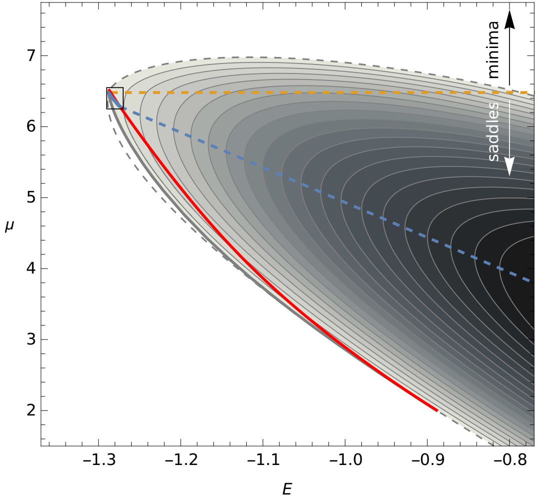

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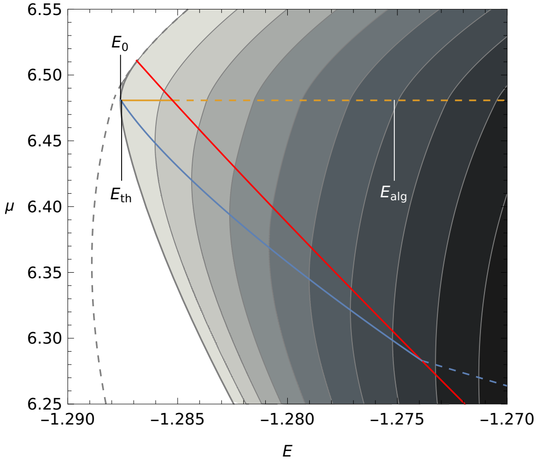

How to count in hierarchical landscapes: full RSB

Example: $2 + 4$ model with full RSB equilibrium.

Quenched complexity predicts full RSB clustering among certain minima and saddles, marginal ground state.

A Crisanti and L Leuzzi, "Spherical $2 + p$ spin-glass model: an exactly solvable model for glass to spin-glass transition", Physical Review Letters 93, 217203 (2004)

JK-D and J Kurchan, "How to count in hierarchical landscapes: a full solution to mean-field complexity", Physical Review E 107, 064111 (2023)

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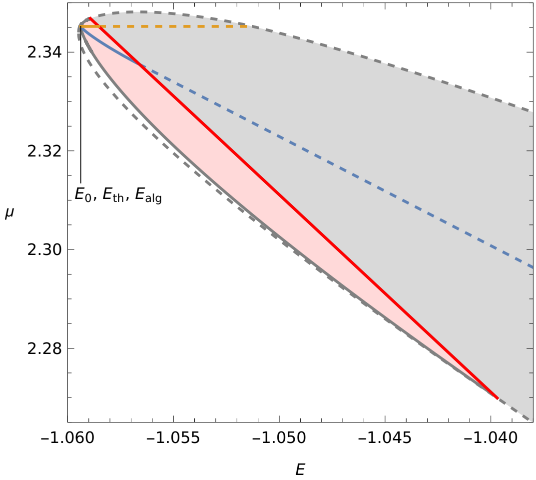


MOUNTAIN PASSES


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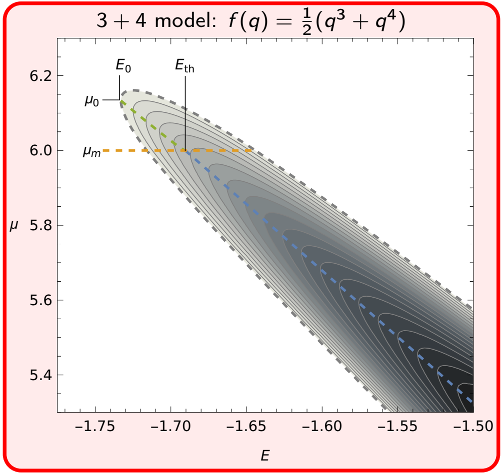
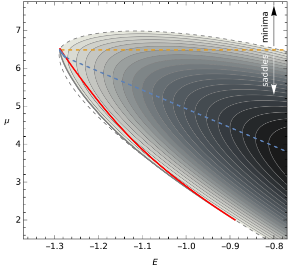
Clustering among saddles

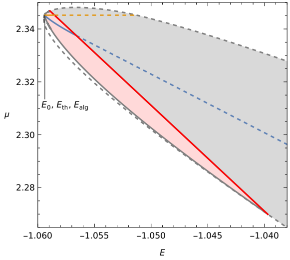


Many models' ground state correctly described by annealed complexity

Quenched complexity shows most clustering among *saddles*

Can RSB arise when equilibrium is trivial?





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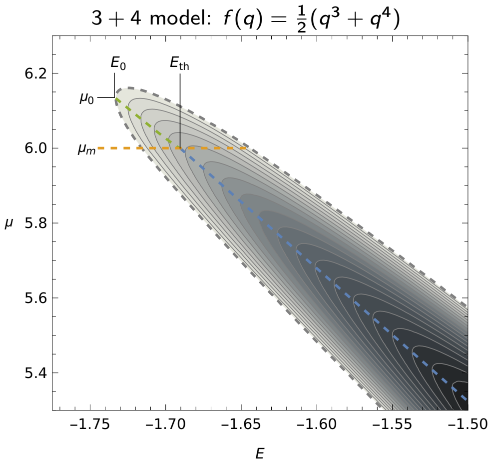

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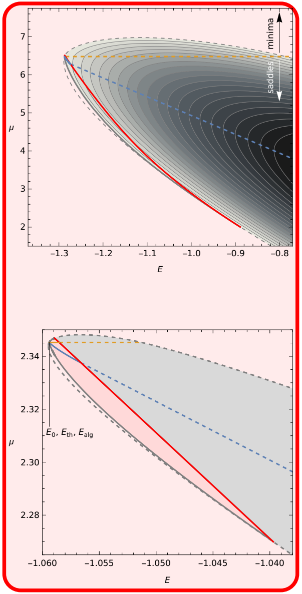

Clustering among saddles

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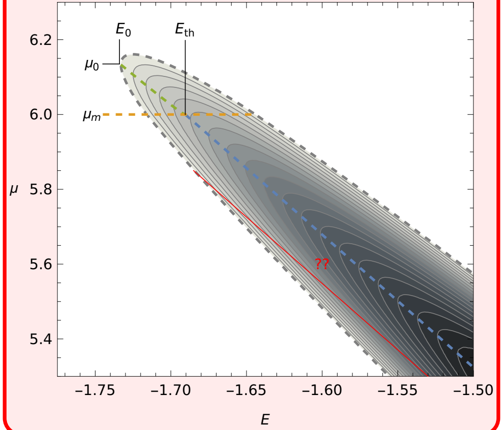

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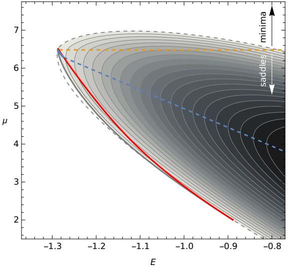
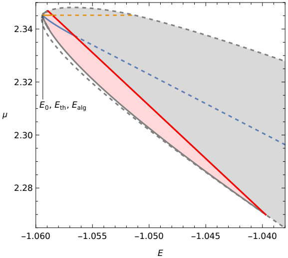




How to find RSB saddles

1RSB complexity has two order parameters:

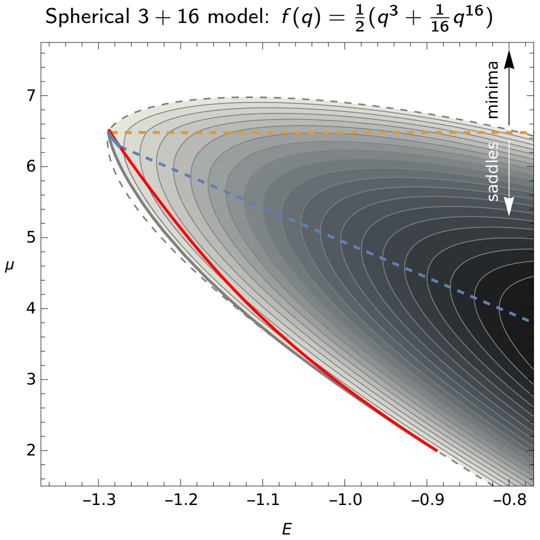
- the tightness of clustering q_1
- the fraction of unclustered pairs x

On red transition line $x = 1$ and $0 < q_1 \leq 1$

At the critical endpoint $x = 1$ and $q_1 = 1$

Can search for critical endpoint from the annealed solution by studying eigenvalues of

$$M = \lim_{x\to 1}\lim_{q_1\to 1}\begin{bmatrix}\frac{\partial^2\mathcal{S}}{\partial q_1^2} & \frac{\partial^2\mathcal{S}}{\partial x\partial q_1}\\ \frac{\partial^2\mathcal{S}}{\partial x\partial q_1} & \frac{\partial^2\mathcal{S}}{\partial x^2}\end{bmatrix}$$



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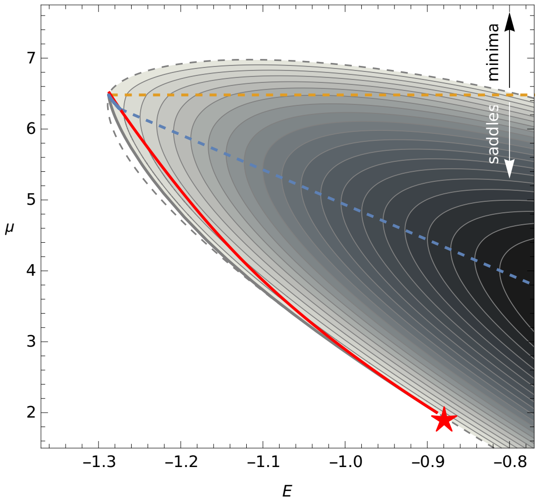
Can search for critical endpoint from the annealed solution by studying eigenvalues of

$$M = \lim_{x\to1} \lim_{q_1\to1} \begin{bmatrix} \frac{\partial^2 \mathcal{S}}{\partial q_1^2} & \frac{\partial^2 \mathcal{S}}{\partial x \partial q_1} \\ \frac{\partial^2 \mathcal{S}}{\partial x \partial q_1} & \frac{\partial^2 \mathcal{S}}{\partial x^2} \end{bmatrix}$$

Spherical 3 + 16 model: $f(q) = \frac{1}{2}(q^3 + \frac{1}{16}q^{16})$



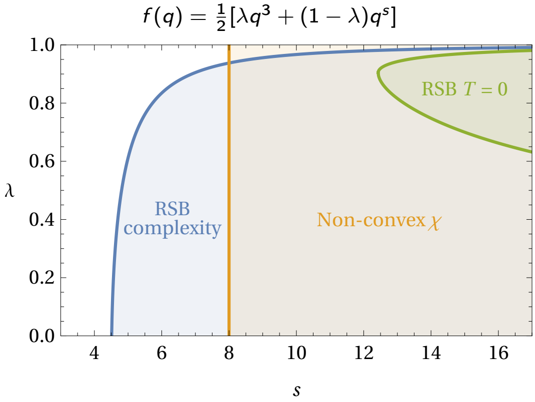
Finding RSB saddles

RSB structure among saddles when $G_f > 0$ for explicit functional G_f

$3+s$ models $f(q) = \frac{1}{2}[\lambda q^3 + (1-\lambda)q^s]$ have a broad range of RSB among saddles

Includes models where clustering among equilibrium states is forbidden (convex $\chi(q) = f''(q)^{-1/2}$)

JK-D, "When is the average number of saddle points typical?", (2023), arXiv:2306.12752v1 [cond-mat.stat-mech]



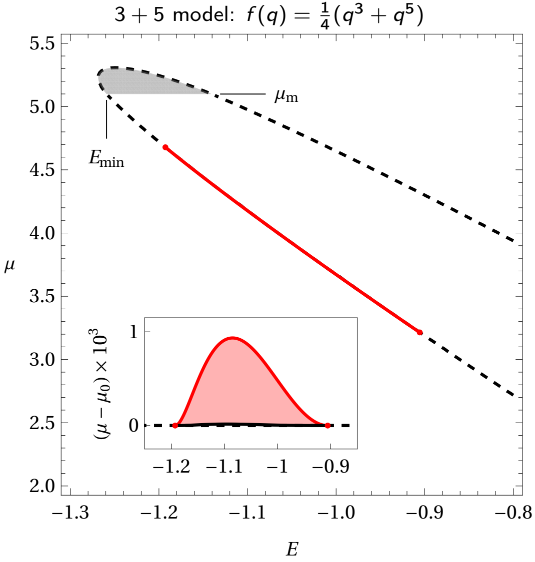
RSB among saddles: example

$3 + 5$ model is forbidden from having clustering between equilibrium states (at most 1RSB equilibrium order)

Wide range of saddles with highest and lowest index show clustering

Implications for emergence of RSB in equilibrium: splitting of states occurs among saddles, not minima

JK-D, "When is the average number of saddle points typical?", (2023), arXiv:2306.12752v1 [cond-mat.stat-mech]



PLAINS


marginal
manifold

Importance of marginal minima

Quench dynamics asymptotically approaches marginal minima

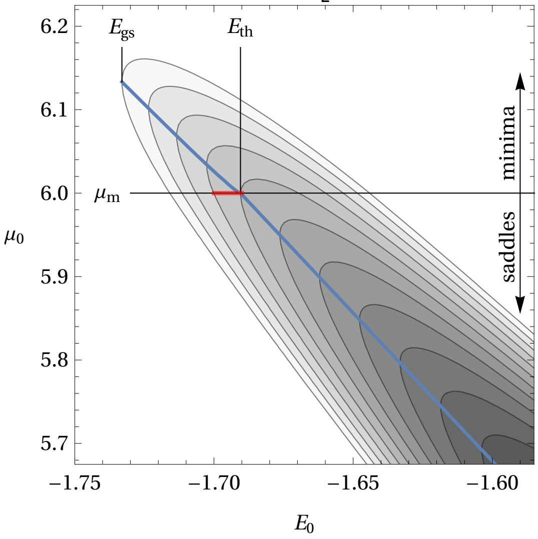
In mixed models, the final energy depends on initial conditions

Threshold energy of Cugliandolo–Kurchan (where most stationary points are marginal) appears unimportant

G Folena, S Franz, and F Ricci-Tersenghi, "Rethinking mean-field glassy dynamics and its relation with the energy landscape: the surprising case of the spherical mixed p-spin model", Physical Review X 10, 031045 (2020)

G Folena and F Zamponi, "On weak ergodicity breaking in mean-field spin glasses", (2023), arXiv:2303.00026v2 [cond-mat.dis-nn]

3 + 4 model: $f(q) = \frac{1}{2}(q^3 + q^4)$

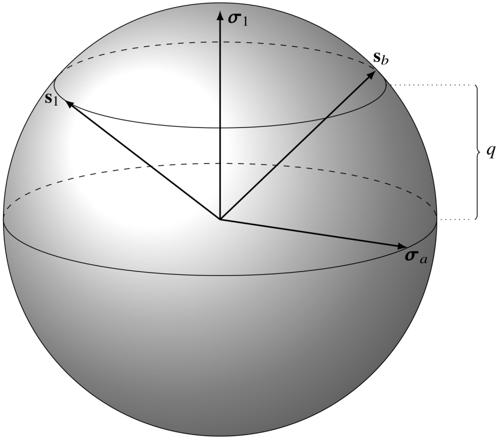


Two-point complexity

Compare different marginal minima by their local neighborhoods: what other stationary points are they nearby?

$$\Sigma_{12} = \frac{1}{N}\int \frac{d\nu(\boldsymbol{\sigma} \mid E_0, \mu_0)}{\int d\nu(\boldsymbol{\sigma}' \mid E_0, \mu_0)} \times \log\left[\int d\nu(\mathbf{s} \mid E_1, \mu_1)\,\delta(Nq - \boldsymbol{\sigma}\cdot\mathbf{s})\right]$$

Gives complexity of stationary points with $(E_1,\ \mu_1)$ constrained at overlap q with a reference point with $(E_0,\ \mu_0)$



Neighborhood of marginal minima

Properties pivot around debunked threshold E_{th}

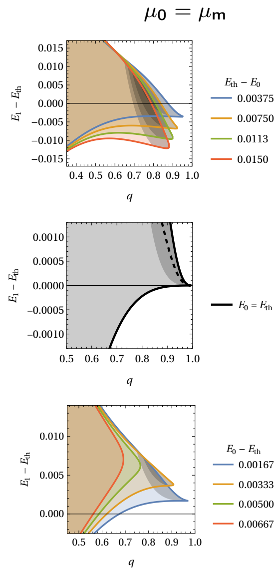
Below E_{th}: Neighbors are distant minima, other marginal minima are distant

At E_{th}: Neighbors are other marginal minima, arbitrarily close together

Above E_{th}: Neighbors are close saddles, other marginal minima are distant

Suggests that typical marginal minima are far apart and separated by high barriers: no 'manifold'

JK-D, "Arrangement of nearby minima and saddles in the mixed spherical energy landscapes", (2023), arXiv:2306.12779v1 [cond-mat.dis-nn]



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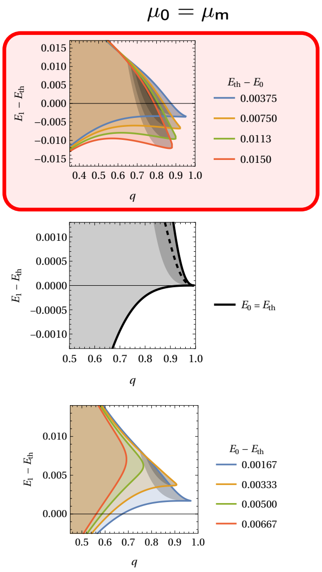

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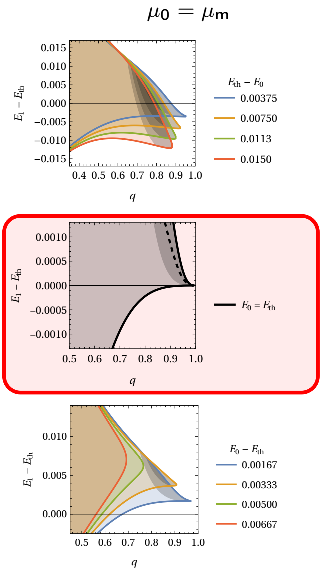

JK-D, "Arrangement of nearby minima and saddles in the mixed spherical energy landscapes", (2023), arXiv:2306.12779v1 [cond-mat.dis-nn]

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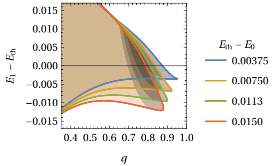
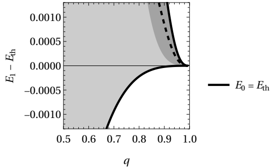
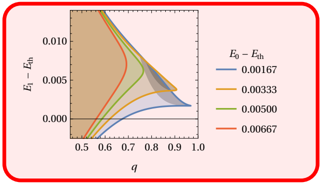
$\mu_0 = \mu_{\text{m}}$







Conclusions

Mixed spherical models have rich geometric structure not present in pure ones

- Clustering of deep minima consistent with hierarchical equilibrium order
- Clustering of saddles without any clustering of minima
- Marginally stable minima without a marginal manifold

JK-D and J Kurchan, "How to count in hierarchical landscapes: a full solution to mean-field complexity", Physical Review E **107**, 064111 (2023)

JK-D, "When is the average number of saddle points typical?", (2023), arXiv:2306.12752v1 [cond-mat.stat-mech]

JK-D, "Arrangement of nearby minima and saddles in the mixed spherical energy landscapes", (2023), arXiv:2306.12779v1 [cond-mat.dis-nn]

Quenched complexity of mean-field models

Details of calculation

$$\overline{\log \mathcal{N}(E,\mu)} = \lim_{n\to 0} \frac{\partial}{\partial n} \overline{\mathcal{N}(E,\mu)^n}$$

$$= \lim_{n\to 0} \frac{\partial}{\partial n} \overline{\int \prod_{a=1}^{n} ds_a \, \delta\big(\nabla H(s_a)\big) \, \big| \det \operatorname{Hess} H(s_a) \big|}$$

$$\overline{\times \delta\big(NE - H(s_a)\big) \, \delta\big(\operatorname{Tr} \operatorname{Hess} H(s_a) - N\mu \big)}$$

$$= \lim_{n\to 0} \frac{\partial}{\partial n} \int \left(\prod_{a=1}^{n} ds_a \right) \overbrace{\overline{\prod_{a=1}^{n} \delta\big(\nabla H(s_a)\big) \, \delta\big(NE - H(s_a)\big)}}^{\text{Function of } H \text{ and } \nabla H \text{ only.}}$$

$$\underbrace{\overline{\times \prod_{a=1}^{n} \big| \det \operatorname{Hess} H(s_a) \big| \, \delta\big(\operatorname{Tr} \operatorname{Hess} H(s_a) - N\mu \big)}}_{\text{Function of Hess } H \text{ only.}}$$

Quenched complexity of mean-field models

Details of calculation

$$\overline{\left|\det \operatorname{Hess} H(s_a)\right| \delta\left(\operatorname{Tr} \operatorname{Hess} H(s_a) - N\mu\right)} \simeq e^{N\mathcal{D}(\mu)} \delta\left(N\mu - s_a \cdot \partial H(s_a)\right)$$

$$\prod_{a=1}^{n} \delta\left(\nabla H(s_a)\right) \delta\left(NE - H(s_a)\right) = \prod_{a=1}^{n} \int d\hat\beta \, d\hat{s}_a \, e^{i\hat{s}_a \cdot \nabla H(s_a) + i\hat\beta\left(NE - H(s_a)\right)}$$

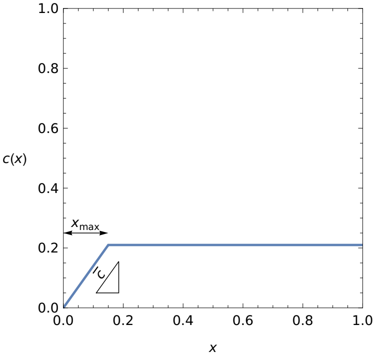
$$C_{ab} = \frac{1}{N} s_a \cdot s_b \qquad R_{ab} = -i\frac{1}{N}\hat{s}_a \cdot s_b \qquad D_{ab} = \frac{1}{N}\hat{s}_a \cdot \hat{s}_b$$

$$\mathcal{S} = \mathcal{D}(\mu) + \hat\beta E - \frac{1}{2}\hat\mu + \lim_{n\to 0}\frac{1}{n}\Bigg(\frac{1}{2}\hat\mu \operatorname{Tr} C - \mu \operatorname{Tr} R$$

$$+\frac{1}{2}\sum_{ab}\left[\hat\beta^2 f(C_{ab}) + (2\hat\beta R_{ab} - D_{ab}) f'(C_{ab}) + R_{ab}^2 f''(C_{ab})\right] + \frac{1}{2}\ln\det\begin{bmatrix} C & iR \\ iR & D \end{bmatrix}\Bigg)$$

Quenched complexity of mean-field models

RS–FRSB transition line



RS–FRSB transition line can be analytically predicted.

1. Treat each function $c(x)$, $r(x)$, $d(x)$ as piecewise linear
2. Substitute into Σ and expand for small $x_{\max}$
3. Look for instability of $x_{\max} = 0$ solution.

$$\mu_{\pm}(E) = \pm\frac{(f'(1)+f''(0))(f'(1)^2 - f(1)(f'(1)+f''(1)))}{(2f(1)-f'(1))f'(1)f''(0)^{1/2}} - \frac{f''(1)-f'(1)}{f'(1)-2f(1)}E$$

Finding RSB saddles

Endpoint (and therefore RSB saddles) exists when $G_f > 0$ for

$$G_f = f' \log \frac{f''}{f'} \left[3y_f(f''-f')f'''-2(f'-2f)f''w_f\right]-2(f''-f')u_f w_f - 2\log^2 \frac{f''}{f'} f'^2 f'' v_f$$

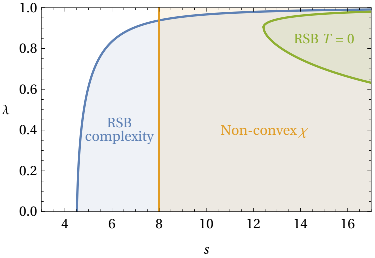
where

$$u_f = f(f' + f'') - f'^2 \qquad v_f = f'(f'' + f''') - f''^2$$

$$w_f = 2f''(f'' - f') + f' f''' \qquad y_f = f'(f' - f) + f'' f$$

For $3 + s$ models with $f(q) = \frac{1}{2}[\lambda q^3 + (1 - \lambda)q^s]$, very broad range have RSB saddles

Includes models where clustering among equilibrium states is forbidden (convex $\chi(q) = f''(q)^{-1/2}$)



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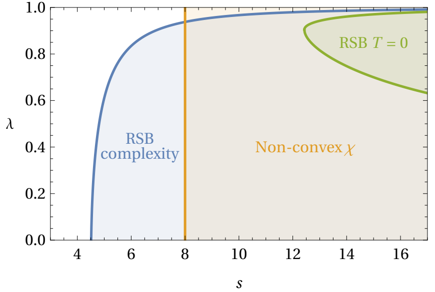
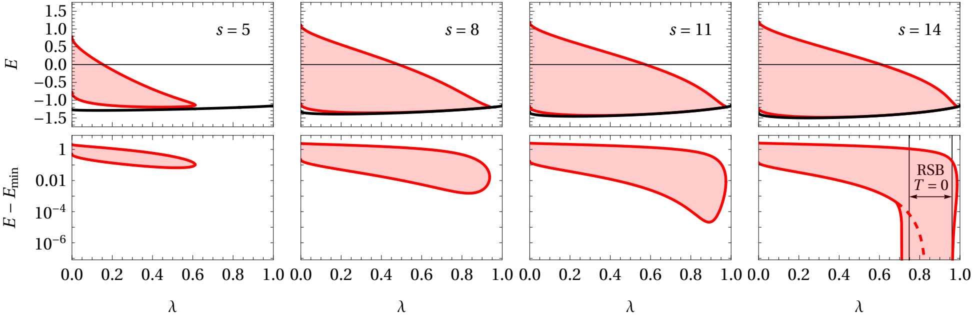
Range of saddle clustering

Clustering among saddles found for most $3 + s$ models with $s \geq 5$ and broad range of energies

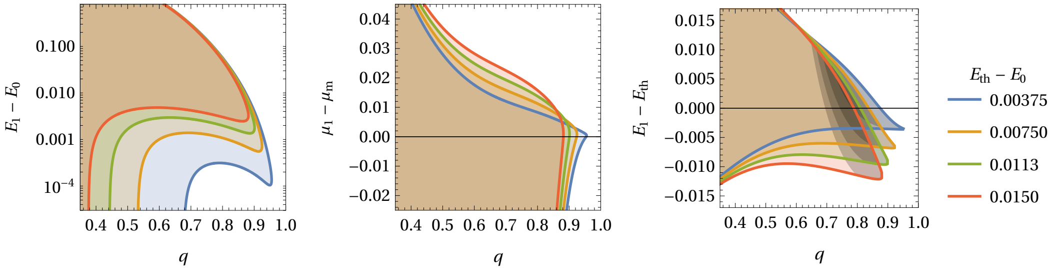
Phase crosses into minima consistent with presence of RSB ground states





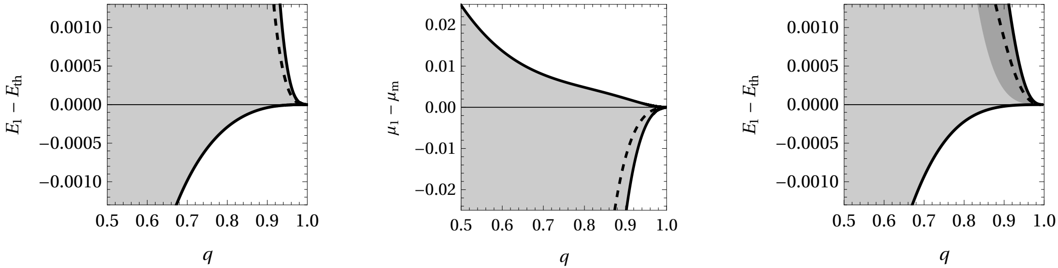
Neighborhood of marginal minima

Below the threshold energy



Neighborhood of marginal minima

At the threshold energy



Neighborhood of marginal minima

Above the threshold energy