Analytic continuation of complex landscapes

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Continuation of phase space integrals is important to define some theories and ameliorate the sign problem in others. By continuing degrees of freedom, such integrals are written as linear combinations of asymptotic series about stationary points of the action, or states. Continuation of parameters preserves these linear combinations, save for the presence of Stokes points which shuffle their contribution.

Complex landscapes have many states, and perhaps many Stokes points, which would make parameter continuation hard. We show that the proliferation (or not) of Stokes points depends on geometric properties of the landscape: 1RSB minima are locally protected, while FRSB ones are not.

Thimble integration

Much of our lives are spent pondering integrals of the form

$$Z(\beta) = \int_M dx\, e^{\beta S(x)} \qquad (1)$$

where S is a function defined on the real manifold M. If S can be continued to a holomorphic function on a complex manifold and M is orientable, this is also a contour integral. The contour can be deformed without changing the value of $Z(\beta)$.

A Lefschetz thimble is the surface reached by gradient descent on $\beta S(x)$ from a state. These are good candidates for contours because, on a thimble,

- Im βS is constant, and salves the sign problem.
- Re βS is bounded from above, and therefore (1) doesn't diverge.

Moreover, Morse theory guarantees that there is a linear combination of thimbles whose antithimbles have the same homology as M relative to the descent. If $\mathcal{J}_\sigma$ is the thimble descending from the state $\sigma \in \Sigma$, then

$$Z(\beta) = \sum_{\sigma\in\Sigma} n_\sigma \int_{\mathcal{J}_\sigma} dz\, e^{\beta S(z)} \qquad (2)$$

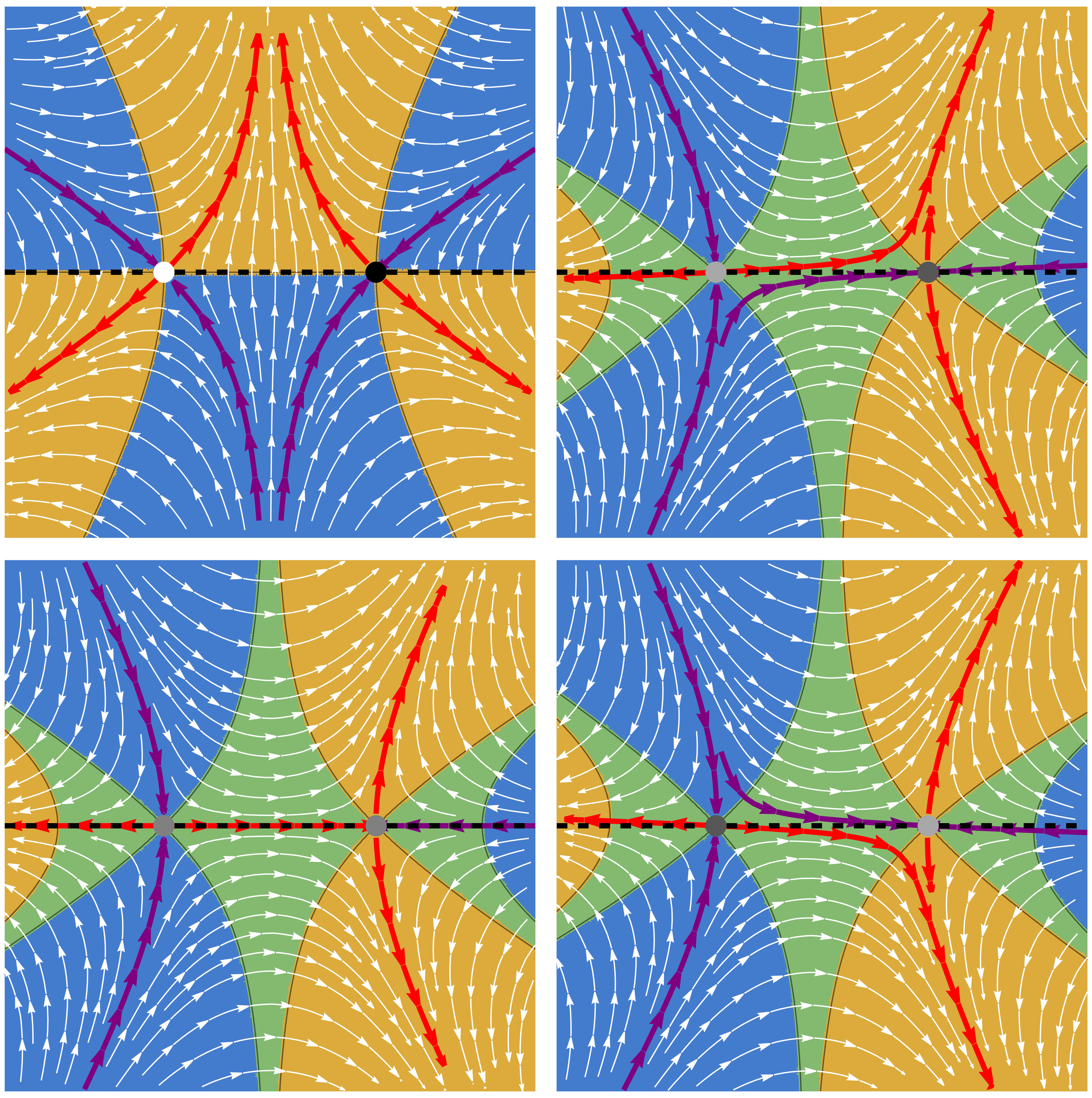

Figure 1: Schematic of thimble integration through a Stokes point for the Airy integral $S(x) = i(x^3/3 - x)$ in the complex x plane. The dashed line shows the contour over the reals. The solid colors are divided by contours of constant Re βS. In order to converge, a contour must begin and end in a yellow region. The dots at $x = \pm 1$ are the two states of S, shaded by Im βS. The arrows depict gradient descent in Re S. The red arrows depict descent out of each state; these are the thimbles. The purple ones depict descent into each state; these are the antithimbles. **Top left:** $\arg\beta = 0$, the (real) Airy integral. The sum of the two antithimbles shares the relative homology of the initial contour. **Top right:** $\arg\beta = \frac{\pi}{2} - \epsilon$. The flow has been smoothly deformed without crossing a Stokes point, and the weights are unchanged. Notice that the original contour no longer corresponds to a convergent integral. **Bottom left:** $\arg\beta = \frac{\pi}{2}$, a Stokes point. The thimbles no longer define contours and care must be taken to evaluate the integral. **Bottom right:** $\arg\beta = \frac{\pi}{2} + \epsilon$. The flow after passing the Stokes point. Now the integral is defined by a contour over only the left thimble.

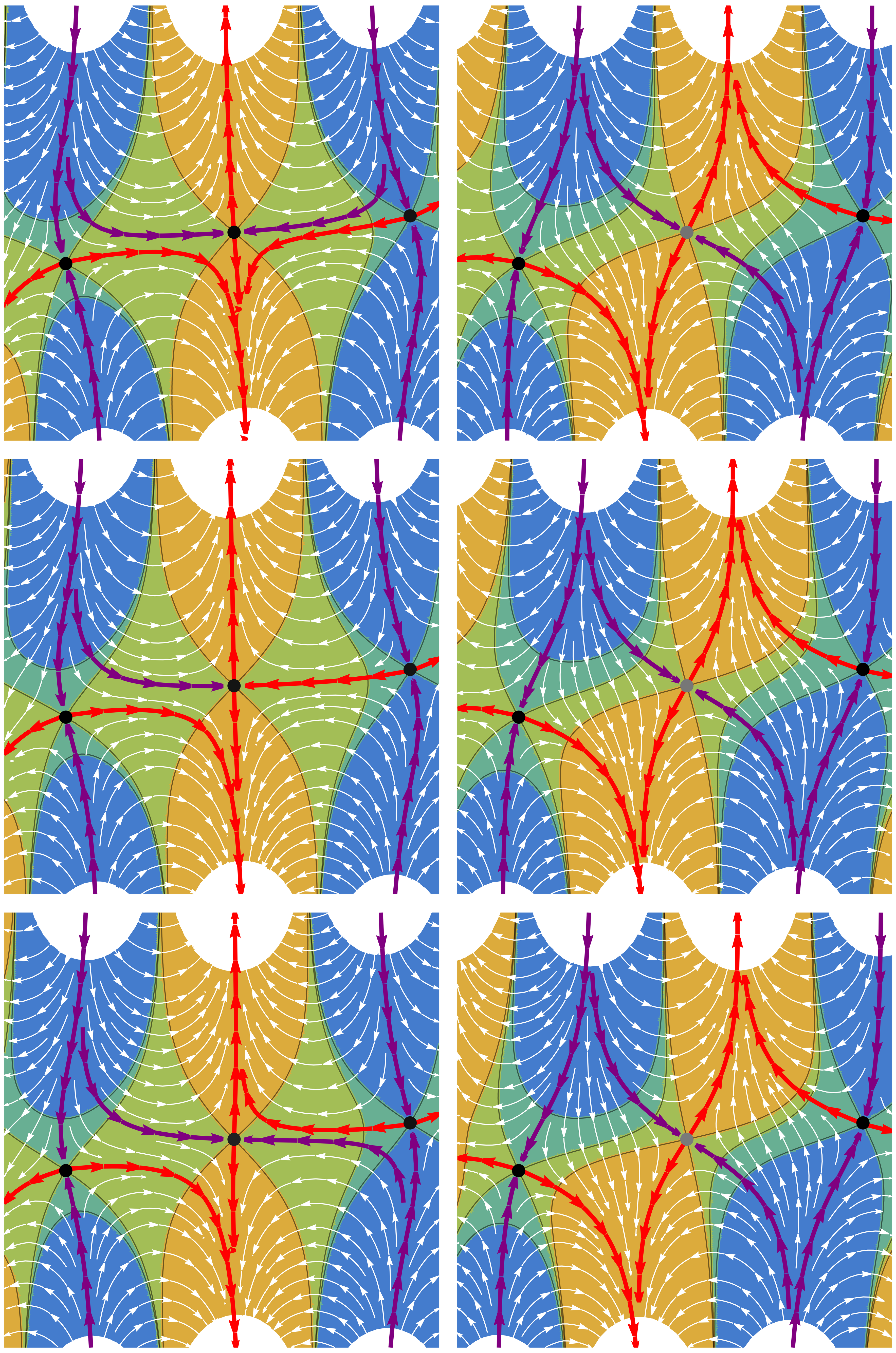

Figure 2: Nearby states brought through the same Im βS by varying the argument of β. **Left, top to bottom:** The coincidence corresponds with a Stokes point. **Right, top to bottom:** The coincidence does not correspond with a Stokes point.

where the n_σ are integer weights fixing the linear combination. Each thimble integral has an asymptotic series in derivatives of S about σ.

Analytic continuation with thimbles

If β is varied, states of S and their thimbles also vary, but their relative homology is usually preserved and the weights unchanged. Under these conditions analytic continuation is simple: the set of integrals and their labels are static and only the argument changes.

The relative homology does change discontinuously at a Stokes point, when two thimbles intersect, forming a Stokes line between them. This can change the integer weights n_σ. Only the point upstream on an descent trajectory can be affected.

Stokes lines in complex landscapes

Two thimbles with the same Im βS share a Stokes line if and only if no other of different Im βS separates them. Each thimble partitions space into two disconnected components. A landscape with many states has many disconnected components, and states that are not neighbors are unlikely to be in the same component and to share Stokes lines.

If a state is gapped, descent on Re βS produces a nonzero expectation for $(\Delta\,\mathrm{Im}\,\beta S)^2$, and Stokes points are rare. If it is gapless, nearby states along marginal directions are also nearby in Im βS, and Stokes points are likely. Continuation of integrals with weight concentrated in gapless states (like low-temperature FRSB) will be hard.