Analytic continuation over complex landscapes

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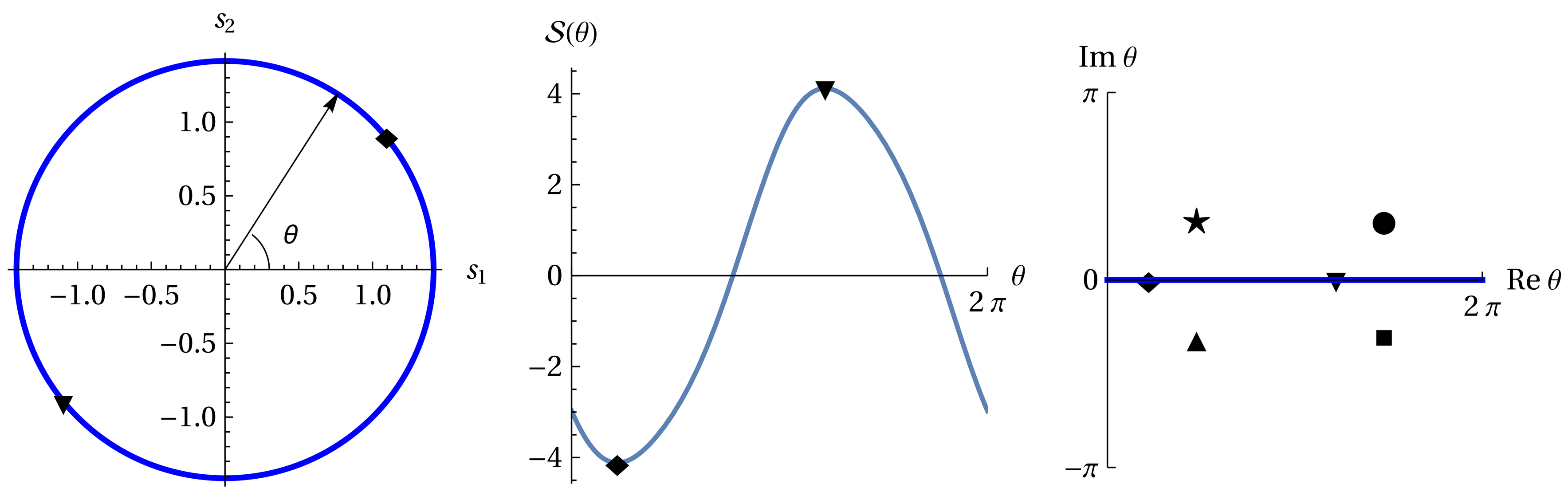

Figure 1: A simple action and its stationary points. **Left:** Phase space of the $N = 2$ spherical (or circular) model, defined for $s \in \mathbb{R}^N$ restricted to the circle $N = s^2$. It can be parameterized by one angle $\theta = \arctan(s_2/s_1)$. Its natural complex extension takes instead $s \in \mathbb{C}^N$ restricted to the hyperbola $N = s^2 = (\operatorname{Re} s)^2 - (\operatorname{Im} s)^2$. The (now complex) angle θ is still a good parameterization of phase space. **Center:** A 3-spin action $\mathcal{S}$. The minimum and maximum are marked with ◆ and ▼, respectively. **Right:** The stationary points of the action in the complex-θ plane. The set of all stationary points is $\Pi = \{$◆, ★, ▲, ▼, ●, ■$\}$, and $\mathcal{N} = 6$ is the number of stationary points.

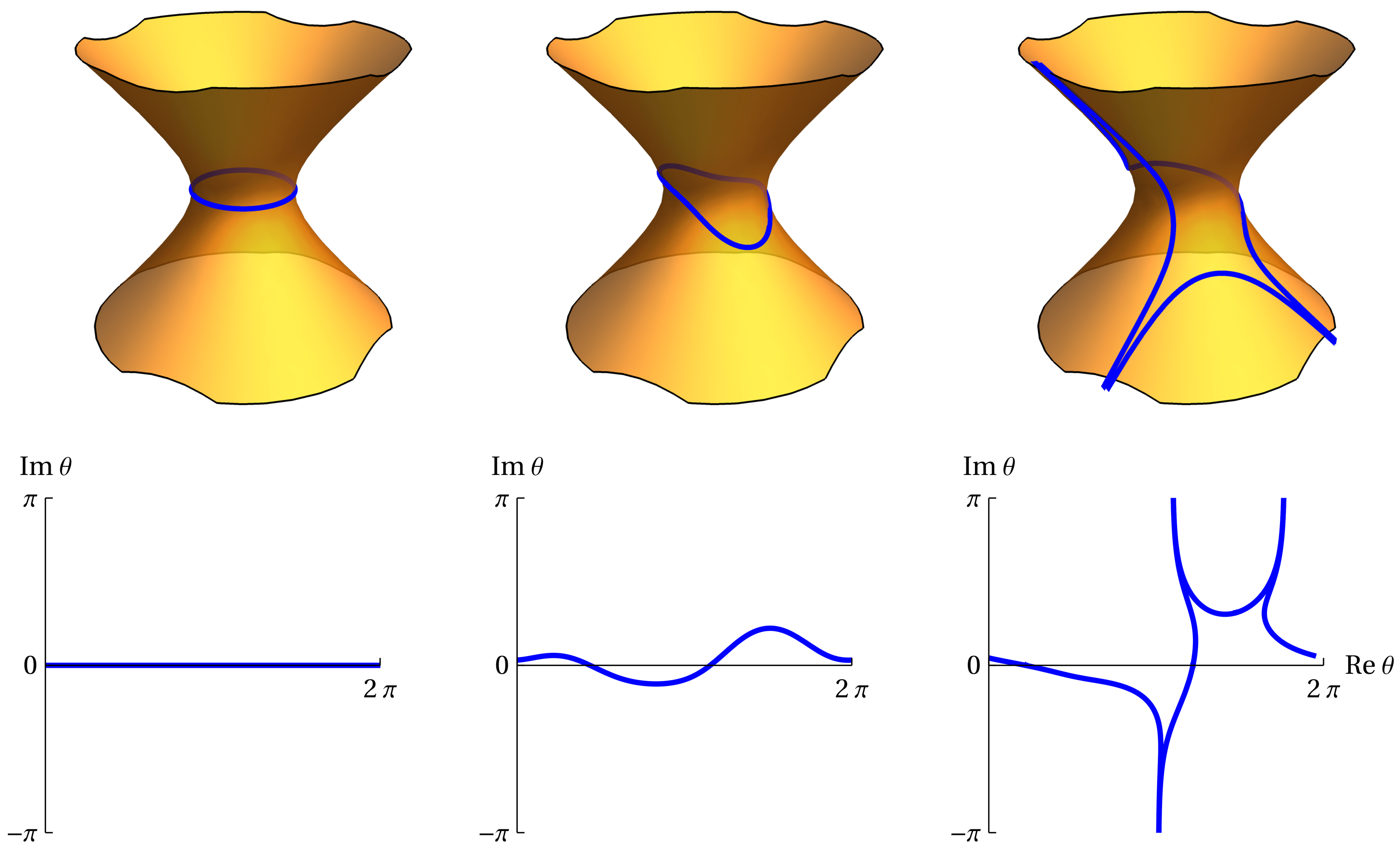

Figure 2: The complex phase space for the circular p-spin model and its standard integration contour. This figure shows the equivalence of the partition function integrals

$$Z = \int_{S^{N-1}} ds\, e^{-\beta\mathcal{S}(s)} = \oint_{\mathcal{C}} ds\, e^{-\beta\mathcal{S}(s)} = \oint_{\mathcal{C}'} ds\, e^{-\beta\mathcal{S}(s)} = \sum_{\sigma\in\Pi} n_\sigma \oint_{\mathcal{J}_\sigma} ds\, e^{-\beta\mathcal{S}(s)} \qquad (1)$$

Left: The phase space integral for the spherical model is equivalent to a contour integral over the contour $\mathcal{C} = S^{N-1}$. **Center:** Complex analysis implies that the contour can be freely deformed into $\mathcal{C}'$ without changing the value of the integral. **Right:** A funny deformation in which pieces have been pinched off to infinity, making the contour a composition of several disconnected "thimble" contours $\mathcal{J}_\sigma$. So long as no poles have been crossed, even this is legal.

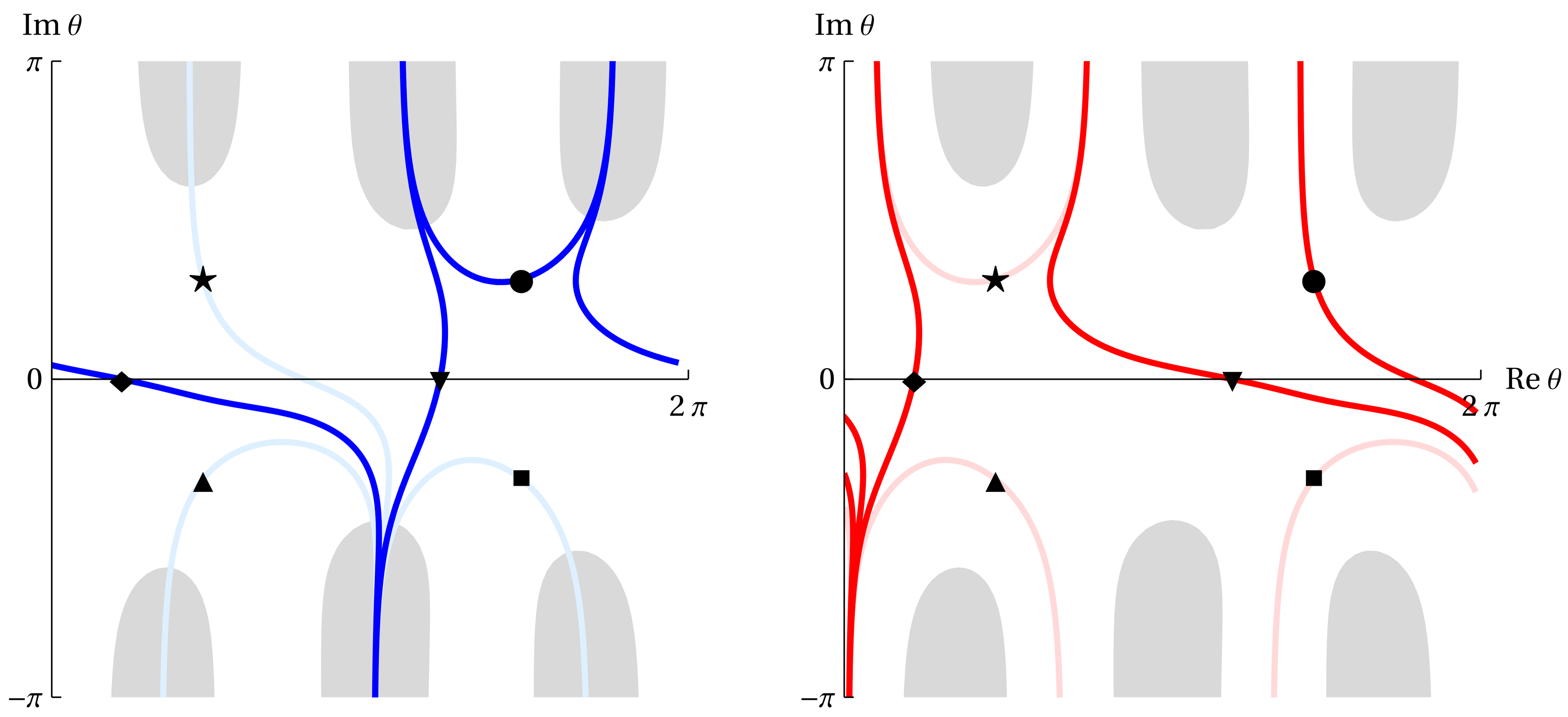

Figure 3: Rules for thimble homology. **Left:** Each stationary point has a thimble, defined by the set of all points which flow into it under gradient descent on $\operatorname{Re}\beta\mathcal{S}$. Thimbles have nice properties as integration contours: they are surfaces of constant phase, they connect good regions of complex phase space where the integrand vanishes (highlighted in gray), and they form a basis for all contours that connect good regions. Here, $\mathcal{C} = \mathcal{J}_◆ + \mathcal{J}_▼ + \mathcal{J}_●$ (the highlighted contour) is homologically equivalent to integration around the circle. **Right:** The antithimbles are defined by all points which flow into the stationary point under gradient *ascent*. Stationary points whose thimbles are involved in the contour have antithimbles that intersect the original contour, the real line.

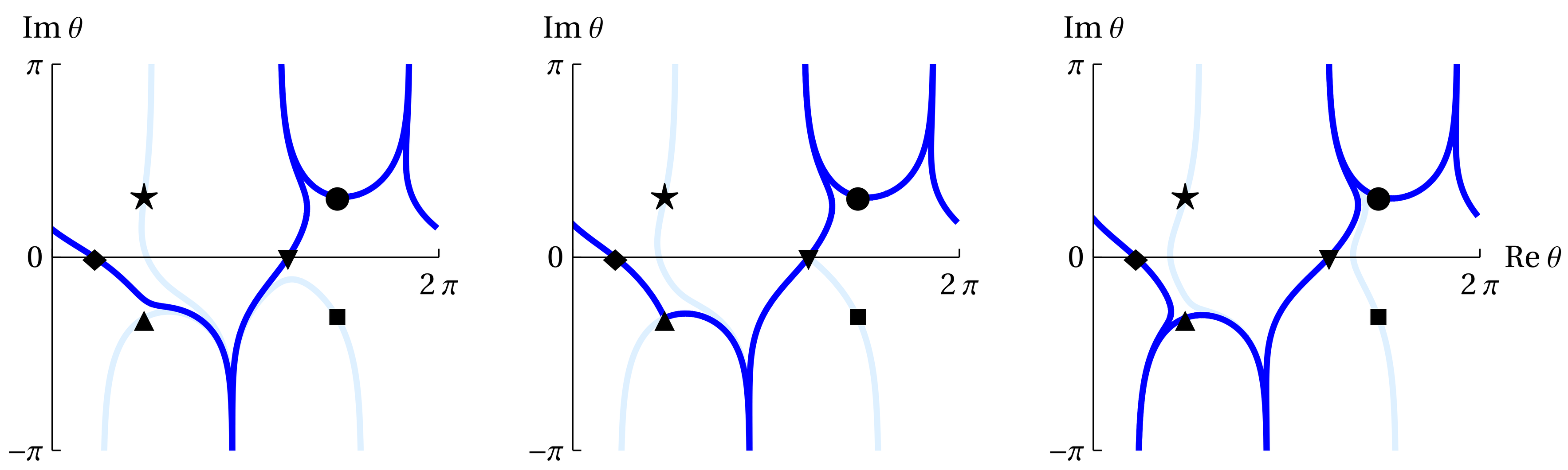

Figure 4: As β (or any another parameter) is continuously varied, the decomposition of the contour into thimbles usually doesn't change. At Stokes points, where two thimbles intersect each other, the contour can suddenly jump. **Left:** The collection of thimbles necessary to progress around the circle from left to right is the same as it was before. **Center:** The thimble $\mathcal{J}_◆$ intersects the stationary point ▲ and its thimble, making the decomposition of the contour into thimbles poorly defined. This is a Stokes point. **Right:** The Stokes point has passed, and the collection of thimbles necessary to produce the path has suddenly changed: now $\mathcal{C} = \mathcal{J}_◆ + \mathcal{J}_▲ + \mathcal{J}_▼ + \mathcal{J}_●$.

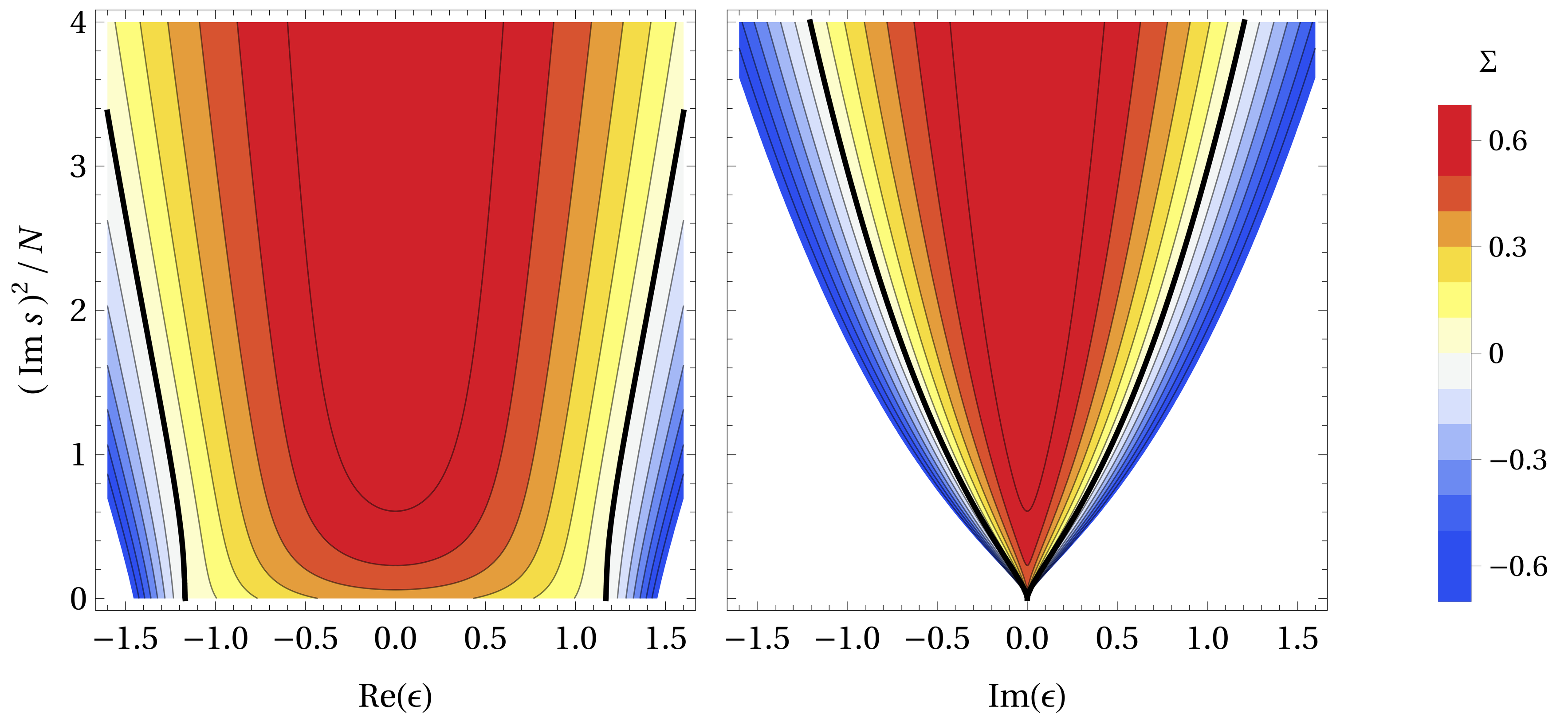

Figure 5: What happens to analytic continuation when the action has a superextensive number of stationary points $\mathcal{N} \sim e^{N\Sigma}$ for large N? First, we examine where the stationary points are in complex phase space. Plots show the complexity Σ of the 3-spin spherical model as a function of energy $\epsilon = \mathcal{S}/N$ and $(\operatorname{Im} s)^2$, the distance into imaginary configuration space. The thick black lines are at zero complexity, the boundary at which stationary points become extremely rare. **Left:** For purely real energy. **Right:** For purely imaginary energy.

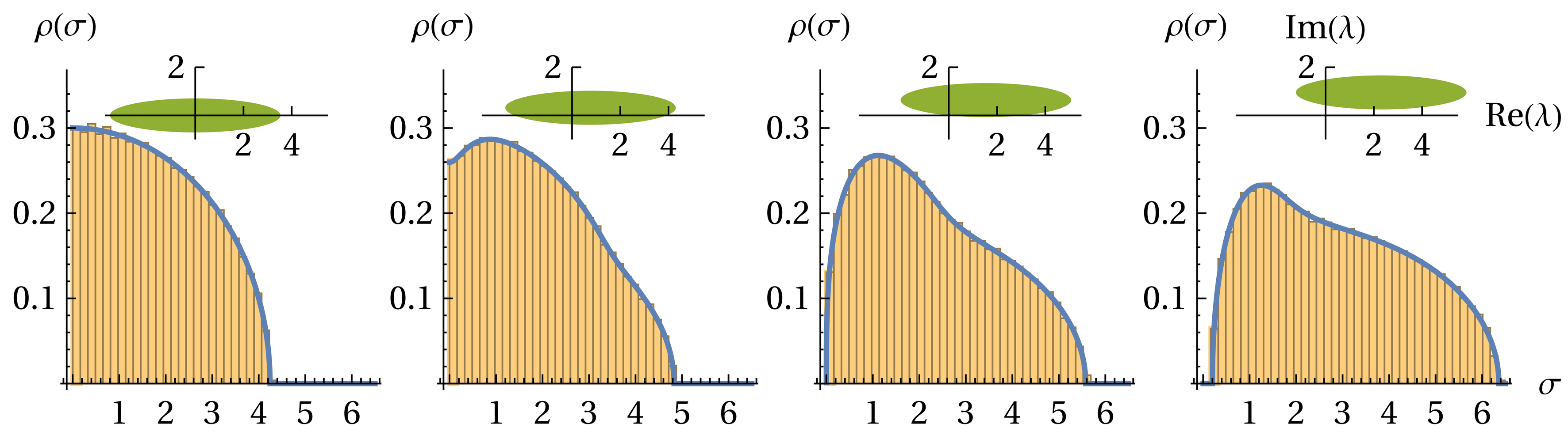

Figure 6: Spectra of the 3-spin spherical model at stationary points with fixed $(\operatorname{Im} s)^2 > 0$ and various energies. The insets show the spectrum of eigenvalues of the hessian $\partial\partial\mathcal{S}$, which is constant inside an ellipse in the complex plane and zero elsewhere. The plots show the spectrum of singular values of the hessian, which correspond to eigenvalues of the *real* part of the hessian, relevant for the thimbles. The energies all have the same complex argument and varying complex magnitude. **Left:** $|\epsilon| = 0$. **Center right:** $|\epsilon| < |\epsilon_{\text{gap}}|$. **Center left:** $|\epsilon| = |\epsilon_{\text{gap}}|$. The boundary of the ellipse intersects the origin, and the singular value spectrum develops a pseudogap. **Right:** $|\epsilon| > |\epsilon_{\text{gap}}|$.

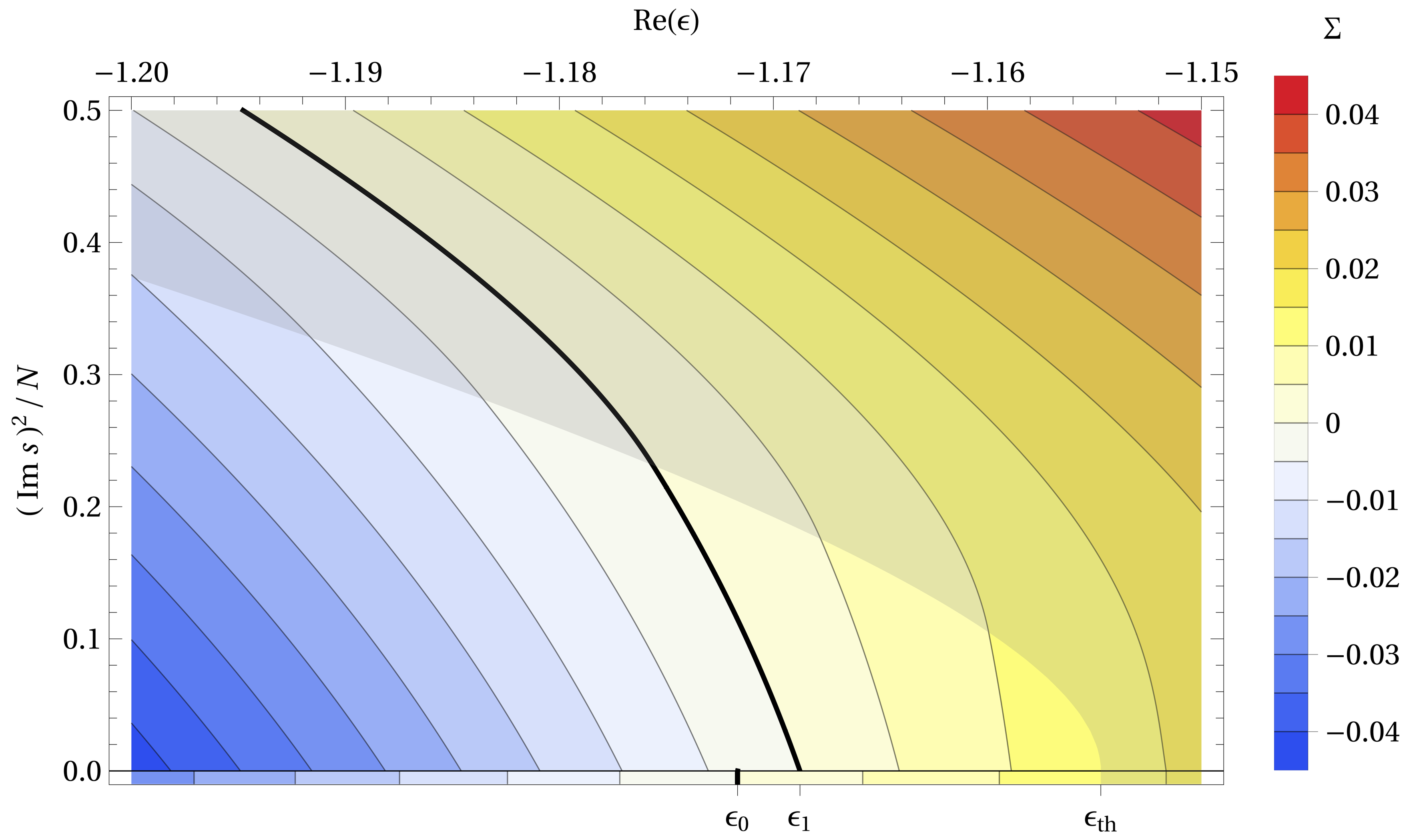

Figure 7: The complexity of the 3-spin spherical model, as in Fig. 5, focused on the lower left. The shaded area shows the region where the spectrum of stationary points is ungapped, whose boundary approaches the threshold energy ϵ_{th} as $\operatorname{Im} s \to 0$. The ϵ_k show the energies at which stationary points with fixed index k vanish in the purely real case: ϵ_0 is the ground state energy, and ϵ_1 is the lowest energy with rank-1 saddles. Below ϵ_{th} the limit $\operatorname{Im} s \to 0$ does not approach the real complexity: complex stationary points in close vicinity to the real plane vanish at ϵ_1, not at ϵ_0 where real stationary points vanish.

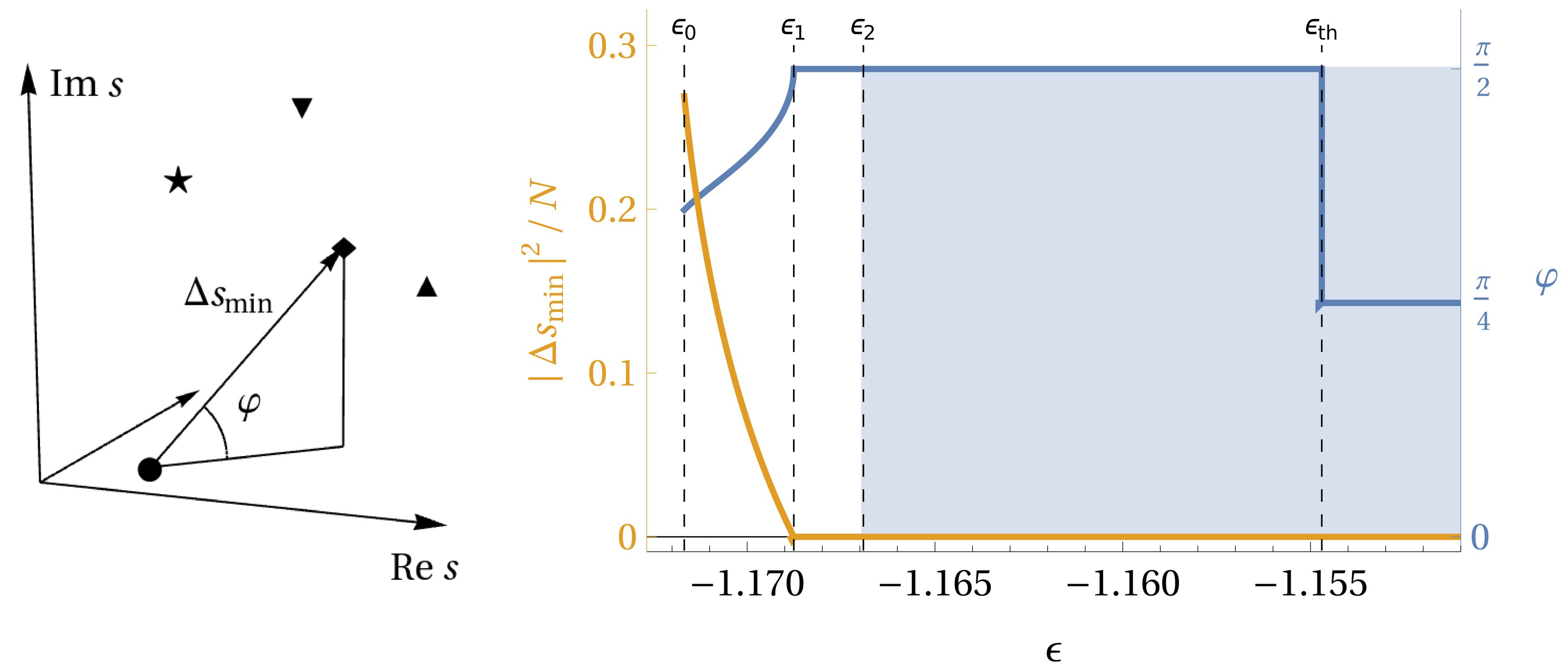

Figure 8: We expect that the stationary points liable to intersect in Stokes points tend to be nearest neighbors. What is the character of these nearest neighbors? **Left:** The "two replica" complexity reveals the population of stationary points like ◆ in the complex plane nearest a given real point ●, as a function of their separation $|\Delta s_{\min}|$ and angle φ made with the real plane. **Right:** Properties of the nearest neighbors as a function of energy. For energies above ϵ_1 there are complex saddles at arbitrarily close distance, while below there is a minimum distance between neighboring saddles. Above ϵ_{th}, the nearest points are broadly distributed at all angles, with the dominant population at 45°. At the threshold the dominant population abruptly shifts to lie at 90°. Below ϵ_2 neighbors are only found at 90°, and below ϵ_1 the angle drifts as the distance between neighbors becomes nonzero. The relative position of nearest stationary points roughly bounds the extent to which analytic continuation can be easily performed.