A-Exam Question #3

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Fracture in quasibrittle or disordered brittle systems—like bone, concrete, and certain rock—has long confounded the traditional tools of material strength analysis. These systems are brittle, like glass, but unlike glass are characterized by a broad distribution of microscopic strengths [1]. This leads to material behavior whose size effects are difficult to predict; strength tests on a concrete sample in the lab have little bearing on the strength of concrete in a dam. The reason for this lies in the fracture procedure. At the tip of a perfect crack in a material, stress diverges. In ordinary brittle materials, this divergent stress leads to immediate breaking[1], while in plastic materials like metals a small region—usually on the order of micrometers—about the crack tip is plastically deformed, absorbing energy and blunting the tip. In quasibrittle materials, however, stress at the crack tip is relieved by microcracking: the formation of myriad small cracks in the vicinity of the larger one, with stress being distributed across all. This microcracking is a highly correlated phenomena, and for materials like concrete takes place on the order of feet [8].

We study fracture using using a simple analogue to elasticity: DC circuits. Stress on a bond is represented by current, and strain at a node is represented by voltage. The scalar-valued objects current and voltage are far more wieldy than their tensor-valued elastic counterparts, and their governing equations match the elastic equations in the limit that strain is purely compressive. Some features are lost in this translation—for instance, we'll talk of percolation on our networks, where the relevant process for elastic system is *rigidity percolation*—but we believe the same scaling laws should apply. The details of our numeric model follow.

A *resistive network* G is a simple connected undirected graph $G = (V, E)$. Each edge $e \in E$ represents a resistor of unit resistance. Assuming $|V| = N$, let $\{x_1, \cdots, x_N\} = V$ and adopt the natural ordering $x_i < x_j$ if $i < j$. In our work we strictly deal with two-dimensional resistive networks, which further requires that G be planar.[2] Kirchoff's law requires that the net current flowing from each node be zero, which (using Ohm's law) means that the voltages $u(x)$ for

[1] This is why you see so few partial cracks in glass.

[2] Or at least planar *locally*. We will be modelling fracture on a torus, after all.

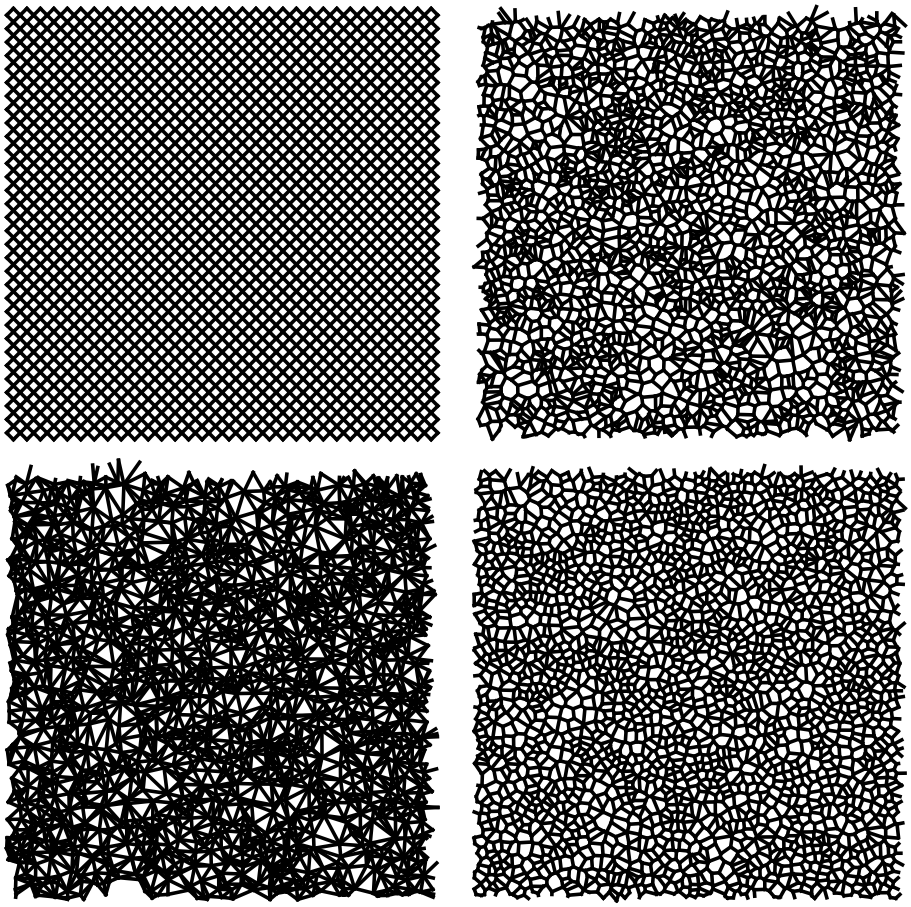

Figure 1: Resistive networks that we use. From left to right, top to bottom: a square network, a network comprised of the edges of voronoi cells for uniform random points, a network comprised of the edges of the Delaney triangulation of uniform random points, and a network comprised of the edges of voronoi cells for hyperuniform random points, generated from the centers of bidisperse jammed particles.

each $x \in V$ obey

$$0 = \sum_{\{y|\{x,y\}\in E\}} u(x) - u(y) \tag{1}$$

We can represent this requirement as a matrix equation. Consider the adjacency function $A(G) : V \times V \to \mathbb{Z}_2$ defined for $x, y \in V$ by

$$(A(G))(x,y) = \begin{cases} 1 & \text{if } \{x,y\} \in E \\ 0 & \text{otherwise} \end{cases}$$

and the degree function $D(G) : V \to \mathbb{N}$ defined for $x \in V$ by

$$(D(G))(x) = \sum_{y \in V} \begin{cases} 1 & \{x,y\} \in E \\ 0 & \text{otherwise} \end{cases}$$

If we define the function $L(G) : V \times V \to \mathbb{Z}$ by $L(G)(x,y) = \delta_{xy}(D(G))(x) - (A(G))(x,y)$, then (1) is equivalent to the requirement that

$$0 = (D(G))(x)u(x) - \sum_{y \in V}(A(G))(x,y)u(y) = \sum_{y \in V}(L(G))(x,y)u(y)$$

which is equivalent to the matrix equation $[L(G)][u] = 0$. $L(G)$ is called the *Laplacian* of G, and $[L(G)]$ is called the *Laplacian matrix*.

Though we have reduced determining vertex voltages to a linear problem, the linear problem is trivial as currently posed. First, the zero vector is obviously a solution. Since we have required that G is connected, $[L(G)]$ has precisely one zero eigenvalue, which corresponds to an eigenvector $[u] = (1, \ldots, 1)$ [3]. All solutions to the problem posed above, then, have $u(x)$ constant for all $x \in V$, and constant voltage implies zero current. This is a complex way of seeing an obvious physical fact: isolated networks of resistors do not spontaneously carry currents. We will need to modify the problem by adding a boundary condition of some kind. Given a resistive network and a boundary condition x of boundary type B, we want to find the current across each edge. The graph $G_\mathrm{B} = (V_\mathrm{B}, E_\mathrm{B})$ and the injective function $f_B : E \to E_\mathrm{B}$ represent a transformed version of G for which the desired boundary has been accounted for[3]. G_B resembles G, but depending on the boundary conditions desired G_B can equal G, contain G, or neither. In all cases, $V_\mathrm{B} \subseteq V$.

No matter the boundary conditions, the same general technique is used. G^* is derived from G based on the boundary type desired. A boundary condition $x : V^* \to \mathbb{R}$, is chosen, along with an additional function $M : V^* \times V^* \to \mathbb{R}$ that helps fix the boundaries in certain cases. A *voltage function* $u : V^* \to \mathbb{R}$ is computed by solving the linear system $[L(G^*) + M(G^*)][u] = [x]$. A *current function* $i : E \to \mathbb{R}$ is then computed from u by the linear transformation m :

[3] This seems very cryptic now, but will become clearer given the coming examples.

$(V^* \to \mathbb{R}) \to (E \to \mathbb{R})$ defined for $\{x, y\} \in E$ and $\{x^*, y^*\} = f^*(\{x, y\}) \in E^*$ by

$$i(\{x, y\}) = (m(u))(\{x, y\}) = \begin{cases} u(x^*) - u(y^*) & x^* > y^* \\ u(y^*) - u(x^*) & x^* < y^* \end{cases}$$

which is the realization of Ohm's law. The resulting values $i(\{x, y\})$ give the current (in the direction $x \to y$ for $x < y$) on the resistor represented by $\{x, y\}$.

The simplest boundaries to adopt are current boundaries. In this case, one defines a *current source function* $I : V^* \to \mathbb{R}$ which gives the net current flowing out of each vertex. Now Kirchoff's law gives

$$I(x) = \sum_{\{y | \{x,y\} \in E^*\}} u(x) - u(y)$$

for $x \in V^*$, or $[L(G^*)][u] = [I]$, which yields nontrivial solutions for any I for which $\sum_{x \in V^*} I(x) = 0$. Mathematically, this is because the sum over vertices of the adjacency function equals the degree, so

$$\begin{aligned} \sum_{x \in V^*} I(x) &= \sum_{x \in V^*} \sum_{y \in V^*} (L(G^*))(x, y) u(y) \\ &= \sum_{y \in V^*} u(y) \big[(D(G^*))(y) - \sum_{x \in V^*} (A(G^*))(x, y) \big] = 0 \end{aligned}$$

but physically this is the statement that the amount of current entering the network must equal that exiting. If one is content selecting point sources within the network as the boundary, one can take $G^* = G$ and $M = 0$ and be done. For our purposes, we will tweak this slightly so as to apply virtual bus-bars to our network. Let $V_\mathrm{t}, V_\mathrm{b} \subset V$ be the vertices on the top and bottom boundaries of our network, respectively. Then we define $V^* = V \cup \{x_\mathrm{t}, x_\mathrm{b}\}$ and

$$E^* = E \cup \big\{ \{y, x_\mathrm{t}\} \mid y \in V_\mathrm{t} \big\} \cup \big\{ \{y, x_\mathrm{b}\} \mid y \in V_\mathrm{b} \big\}$$

Let $I(x) = 0$ for all $x \in V$ and $I(x_\mathrm{t}) = 1$, $I(x_\mathrm{b}) = -1$. We have added two fictitious vertices to our network, connected them to the network's top and bottom, and made them a source and sink, respectively. In order to produce more numeric stability and make voltage solutions unique, we will also require $u(x_\mathrm{t}) = 0$ by setting $M(x, y) = \delta_{x_\mathrm{t} x} \delta_{x_\mathrm{t} y}$, as

$$\begin{aligned} 0 &= I(x_\mathrm{t}) - \sum_{y \in V^*} \big[(L(G^*))(x_\mathrm{t}, y) + M(x_\mathrm{t}, y) \big] u(y) \\ &= u(x_\mathrm{t}) + I(x_\mathrm{t}) - \sum_{y \in V^*} (L(G^*))(x_\mathrm{t}, y) u(y) = u(x_\mathrm{t}) \end{aligned}$$

This removes the zero eigenvalue from $[L(G^*) + M]$ and therefore the degree of freedom represented by constant shifts in u. In order for the presence of

the boundary to minimally effect the resulting currents, we often set the resistance of the extra edges in $E^* - E$ to near-zero values, which is equivalent to multiplying all associated terms in the Laplacian by a large constant.

Now suppose we want to fix the voltage on a subset of vertices $V_1 \subset V$ to equal U and the voltage on another subset $V_0 \subset V$ to equal zero. These vertices will be removed from the linear problem because they no longer need to be solved for, but how do we account for them in the solution for the free vertices? If we define

$$E_0(x) = \{\{x, y\} \in E \mid x \in V_0\} \qquad E_1(x) = \{\{x, y\} \in E \mid x \in V_1\}$$
$$E_0 = \bigcup_{x \in V} E_0(x) \qquad E_1 = \bigcup_{x \in V} E_1(x)$$

and $E^* = E - (E_0 \cup E_1)$, then for a vertex $x \in V^* = V - (V_0 \cup V_1)$, we have

$$\begin{aligned} 0 &= \sum_{\{x,y\} \in E} (u(x) - u(y)) = \sum_{\{x,y\} \in E^*} (u(x) - u(y)) + \sum_{\{x,y\} \in E_0} (u(x) - u(y)) + \sum_{\{x,y\} \in E_1} (u(x) - u(y)) \\ &= \sum_{y \in V^*} L_{G^*}(x, y) u(y) + u(x) \Big(\sum_{\{x,y\} \in E_0} 1 + \sum_{\{x,y\} \in E_1} 1 \Big) - \sum_{\{x,y\} \in E_1} U \end{aligned}$$

If we write $M(x, y) = \delta_{xy}(|E_0(x)| + |E_1(x)|)$, then we have

$$U|E_1(x)| = \sum_{y \in V^*} (L_{G^*}(x, y) + M(x, y))y$$

Thus the boundary condition x is given by $x(v) = E|E_1(v)|$ for all $v \in V^*$.

If G is locally planar but globally a tiling of a torus, we can still compute currents on it assuming that voltage around one axis of the torus increases by ΔU with each traversal[4] This would require the voltage function to be multivalued, and actually violate Kirchoff's laws. We will get around this by cutting the torus. First, cut the torus along one axis. Let the set of cut edges be E_c, and let V_1 and V_2 be the sets of vertices represented in E_c on each side of the cut. Consider new sets V_1^* and V_2^* with $|V_1^*| = |V_1|$ and $|V_2^*| = |V_2|$, and let $f_1 : V_1 \to V_1^*$ and $f_2 : V_2 \to V_2^*$ be bijections. Then define the graph $G_f = (V_f, E_f)$ where $V_f = V \cup V_1^* \cup V_2^*$,

$$E_1 = \big\{\{x, f_1(y)\} \mid \{x, y\} \in E_c, y \in V_1\big\} \quad E_2 = \big\{\{x, f_2(y)\} \mid \{x, y\} \in E_c, y \in V_2\big\}$$

and

$$E_f = (E - E_c) \cup E_1 \cup E_2$$

Kirchoff's laws will faithfully reproduce the torus current on this graph provided we ensure that $u(f_1(y)) = u(y) + \Delta U$ and $u(f_2(y)) = u(y) - \Delta U$. Let $E' =$

[4]This is equivalent to there being some net magnetic flux through the center of the torus.

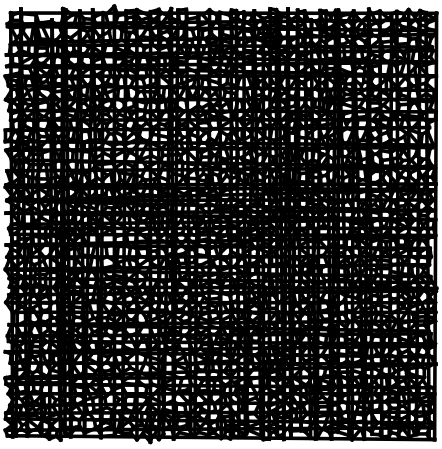

Figure 2: While all figures in this paper seem to show planar networks, they are actually just visually appealing representations of networks on a torus. This is a hyperuniform voronoi network with the wrapping edges explicitly shown—not very comprehensible!

$E - E_c$. We have, for some $x \in V$,

$$\begin{aligned}
0 = \sum_{\{x,y\}\in E_f} (u(x) - u(y)) &= \sum_{\{x,y\}\in E'} (u(x) - u(y)) + \sum_{\{x,y\}\in E_1} (u(x) - u(y)) + \sum_{\{x,y\}\in E_2} (u(x) - u(y)) \\
&= \sum_{y\in V} L_{(V,E')}(x,y)u(y) + \sum_{\{x,y\}\in E_1} (u(x) - u(f_1^{-1}(y)) + \Delta U) + \sum_{\{x,y\}\in E_2} (u(x) - u(f_2^{-1}(y)) - \Delta U) \\
&= \sum_{y\in V} L_{(V,E')}(x,y)u(y) + \sum_{\{x,y\}\in E_c} (u(x) - u(y)) + \sum_{\{x,y\}\in E_1} \Delta U - \sum_{\{x,y\}\in E_2} \Delta U \\
&= \sum_{y\in V} L_G(x,y) + |E_1(x)|\Delta U - |E_2(x)|\Delta U
\end{aligned}$$

Remarkably, the linear problem can be solved in terms of the Laplacian of the original torus, with specially added sources and sinks! We let $x(v) = (|E_2(v)| - |E_1(v)|)\Delta U$. We can now also remove a final trivial degree of freedom by setting $u(x_1) = 0$ as we did for the current boundaries.

A *fuse network* $F = (G, t)$ is a resistive network $G = (V, E)$ with a function $t : E \to (0, 1]$ that corresponds to the breaking current of each fuse [5]. A fuse network is fractured deterministically under the following procedure. The current function $i : E \to \mathbb{R}$ is computed on G using whatever boundaries are desired. The edge $e \in E$ which maximizes $|i(e)|/t(e)$ is found. A new graph $G' = (V, E - \{e\})$ and a new fuse network $F' = (G', t')$ (for the natural restriction of the domain of t) are then created. This simulates the breaking of elements in the network as the voltage (or current) is increased adiabatically. When this procedure is iterated, it results in a chain $(F, F', F'', \ldots, F^{\prime\cdots\prime})$ of networks, the

final one of which is *fractured*. A network is fractured when

- (in the case of current boundaries) the source and sink belong to different connected clusters, or
- (in the case of voltage or toriodal boundaries) the current function $i(e) = 0$ for all $e \in E$.

The distinction in the definition of "fractured" is present only because in the current case the linear problem used to find the current function has no solution, and so it cannot be solved for $i(e) = 0$. This process of breaking networks in sequence can be done very efficiently. Because, when a fuse is broken, only four elements in two rows and columns of the Laplacian are affected, one can solve for the currents of the new graph by using a rank-one transformation to the Laplacian's Cholesky decomposition (used by the numeric solver) rather than refactor the new Laplacian each time, a very expensive operation [6]. We use the CHOLMOD library for this [2].

For a given network, one can find the conductivity σ of the network (the analogue to elastic modulus) by

- (in the case of current boundaries) dividing the applied current (usually unit current) by the voltage difference between the source and sink, or
- (in the case of voltage boundaries) identifying a minimal set of edges which cuts the system, summing the currents on these edges *in the direction they cross the cut*, and dividing it by the applied voltage (usually unit voltage).

For the chain of fuse networks defined above, the conductivity should be a monotonically decreasing function. One network in the chain will be known as the *critical fuse network*. If e is the edge broken in each network and I is the current flowing across the network, then the critical network is that for which $It(e)/|i(e)|$. In the case of current boundaries $I = 1$, and in the case of voltage boundaries $I = \sigma$. The quantity $It(e)/|i(e)|$ can be thought of as the current on the network *in the units of the fuse threshold*—put another way, if we were to actually simulate slowly ramping up the boundary conditions until the literal result of our linear problem equals the fuse value, this would be the value of the applied current. This has other significance: if the value of $It(e)/|i(e)|$ for a network in the chain is greater than that of n networks following it, then the breaking of the fuse in the first network can be thought of as having caused the breaking of those in the next n networks—an avalanche. Another way of viewing the critical network, then, is the network prior to the final avalance that breaks the system.

In our work, $t(e)$ for each $e \in E$ is taken from the distribution $\beta x^{\beta-1}$. The parameter β allows us to tune the disorder of our networks. Previous work has demonstrated, numerically, the validity of a simple scaling theory for finite-size effects based on the following arguments. Small β corresponds to high disorder, where material strengths vary across orders of magnitude, while large

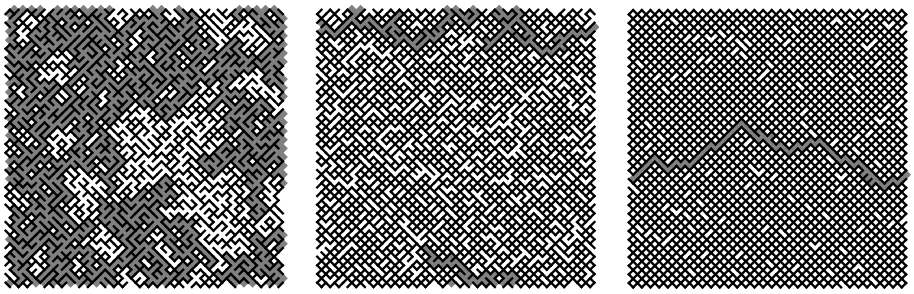

Figure 3: Broken networks for three choices of β (from left to right): 0.03, 0.5, and 3. The fracture surface is highlighted in each.

β corresponds to very little disorder. For any β, the expected value of the nth strongest fuse can be shown to be

$$\langle x_n \,|\, N, \beta \rangle = \frac{\Gamma(N+1)\Gamma(n+\frac{1}{\beta})}{\Gamma(N+1+\frac{1}{\beta})\Gamma(n)} = (\tfrac{n}{N})^{1/\beta} + \mathcal{O}(N^{-1})$$

For sufficiently small β, the ratio $((n+1)/n)^{1/\beta}$ of any two subsequent microscopic strengths dwarfs the ratios of stress in those regions, and regions break simply in the order of their strengths. Since strengths are independent, this corresponds to breaking regions in random order, with the system fractured with a contiguous broken region spans the system.[5] This is simply a percolation problem. Therefore one expects that measured quantities in the limit of small β will scale as they do in percolation, which is a renormalization group (RG) fixed point. The disorder parameter β appears as a relevant perturbation from this fixed point, since fracture quickly becomes non-percolation-like as β is increased.

Suppose that, in the vicinity of the percolation fixed point, the disorder parameter β scales as $\frac{\mathrm{d}\beta}{\mathrm{d}\ell} = \alpha\beta$. We can incorporate the affect of β into the standard percolation scaling theory [9]. Consider the probability distribution $P(p \,|\, L, \beta)$ that a given system percolates (or is broken) at bond fraction p given system size L and disorder parameter β. We know that this function should behave, under rescaling by length b, like

$$P(p \,|\, L, \beta) = f(p - p_c, L, \beta) = b^{1/\nu} f\big(b^{1/\nu}(p - p_c), b^{-1}L, b^{\alpha}\beta\big)$$

where ν is known from percolation theory. Rescaling sufficiently much so that $b^n = L$, we have

$$P(p \,|\, L, \beta) = L^{1/\nu} f\big(L^{1/\nu}(p - p_c), 1, L^{\alpha}\beta\big) = L^{1/\nu} \mathcal{P}\big(L^{1/\nu}(p - p_c), L^{\alpha}\beta\big)$$

[5] One could also study the case where the system is considered fractured when its rigidity goes to zero. Our scaling analysis would be unchanged, with the percolation exponents substituted for those of rigidity percolation, though our numerics would need to be changed entirely.

where we have defined $\mathcal{P}(X,Y) \equiv f(X,1,Y)$. The scaling theory of percolation also indicates that the probability distribution $n_c(s \mid p)$ of cluster sizes in the percolating system given the bond fraction p is

$$n_c(s \mid p) = s^{-\tau_c} g\big(s^{\sigma}(p - p_c), s^{-\Omega} u\big)$$

where u is the leading irrelevant scaling variable in percolation and τ_c, σ, and Ω are known from percolation theory. It follows that the distribution of cluster sizes given system size L and disorder β scales like

$$\begin{aligned}
n_c(s \mid L, \beta) &= \int_0^1 n_c(s \mid p) P(p \mid L, \beta) \, \mathrm{d}p \\
&= s^{-\tau_c} L^{1/\nu} \int_0^1 f\big(s^{\sigma}(p - p_c), s^{-\Omega} u\big) \mathcal{P}\left(L^{1/\nu}(p - p_c), L^{\alpha}\beta\right) \mathrm{d}p \\
&= s^{-\tau_c} L^{1/\nu} \int_{-p_c}^{1-p_c} f(s^{\sigma} p', s^{-\Omega} u) \mathcal{P}(L^{1/\nu} p', L^{\alpha}\beta) \, \mathrm{d}p' \\
&= s^{-\tau_c} \int_{-L^{1/\nu} p_c}^{L^{1/\nu}(1-p_c)} f(L^{-1/\nu} s^{\sigma} x, s^{-\Omega} u) \mathcal{P}(x, L^{\alpha}\beta) \, \mathrm{d}x \\
&\simeq s^{-\tau_c} \int_{-\infty}^{\infty} f(L^{-1/\nu} s^{\sigma} x, s^{-\Omega} u) \mathcal{P}(x, L^{\alpha}\beta) \, \mathrm{d}x \\
&= s^{-\tau_c} \mathcal{N}_c(L^{-1/\nu\sigma} s, L^{\alpha}\beta, L^{-\Omega/\nu\sigma} u) = s^{-\tau_c} \mathcal{N}_c(L^{-d_c} s, L^{\alpha}\beta, L^{-\omega} u)
\end{aligned}$$

where we have made the approximation that $P(x, L^{\alpha}\beta)$ is sharply peaked around $x = 0$ and have defined

$$\mathcal{N}_c(X, Y, Z) \equiv \int_{-\infty}^{\infty} f(X^{\sigma} x, X^{-\Omega} Z) \mathcal{P}(x, Y) \, \mathrm{d}x$$

We have also identified $d_c = 1/\nu\sigma$, the fractal dimension of percolation clusters, and $\omega = \Omega/\nu\sigma$. The nth moment of this distribution is

$$\begin{aligned}
\langle s^n \rangle_c &= \int_0^{\infty} s^n n(s \mid L, \beta) \, \mathrm{d}s = \int_0^{\infty} s^{n-\tau_c} \mathcal{N}_c(L^{-d_c} s, L^{\alpha}\beta, L^{-\omega} u) \, \mathrm{d}s \\
&= L^{d_c(n+1-\tau_c)} \int_0^{\infty} x^{n-\tau} \mathcal{N}_c(x, L^{\alpha}\beta, L^{-\omega} u) \, \mathrm{d}s \\
&= L^{d_c(n+1-\tau_c)} \mathcal{N}_c^{(n)}(L^{\alpha}\beta, L^{-\omega} u)
\end{aligned}$$

These predictions are bourne out by our numerics (Fig. 5) which shown the cluster exponents for two-dimensions well-explain the scaling of the empirical cluster size distribution.

This theory further consists of the following observation: for intermediate values of β, percolation-like behavior seems to continuously give way to crack nucleation by precursor avalanches, in which the failure of one regions triggers the failure of others without increasing the stress. These avalanches do not appear in either the small or large β limits; in the former case, strengths are

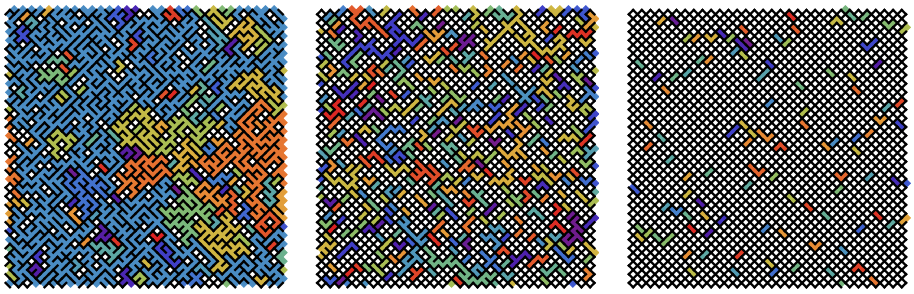

Figure 4: Critical networks for three choices of β (from left to right): 0.03, 0.5, and 3. The nontrivial connected clusters are shown in each. Clusters are defined by connected components on the resistive network's dual.

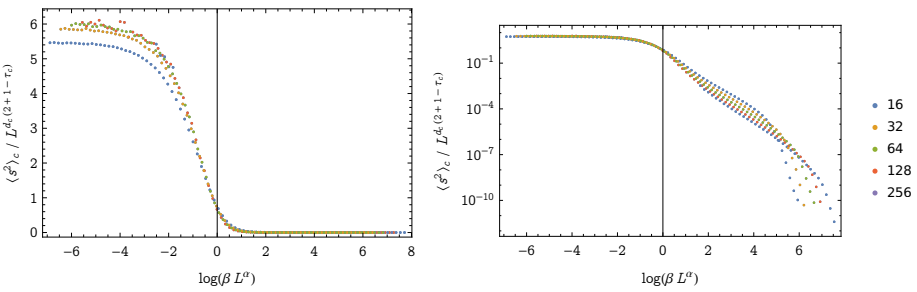

Figure 5: Plots of the second moment of the cluster size distribution normalized by scaling predictions, both single-log and log–log.

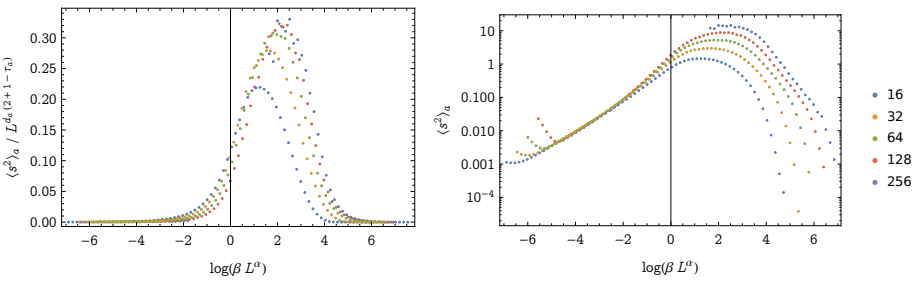

Figure 6: Plots of the second moment of the avalanche size distribution normalized by scaling predictions, both single-log and log–log.

Exponent	2D value
ν	$\frac{4}{3}$
τ_c	$\frac{187}{91}$
σ	$\frac{36}{91}$
ω	$\frac{3}{2}$ [10]
d_c	$\frac{91}{48}$

Table 1: Percolation exponents and their two-dimensional values.

too separated to trigger more than one to fracture at a time at fixed stress, and in the other the mean of the avalanche size distribution has grown so large that the first bond broken is likely to trigger an avalanche that immediately severs the system. We will assume a scaling form for the avalanche size distribution $n_a(s)$, or

$$n_a(s \mid L, \beta) = s^{-\tau_a} \mathcal{N}_a(L^{-d_a} s, L^\alpha \beta, L^{-\omega} u)$$

where we have assumed that the avalanche size distribution corrected by the same nonanalytic correction to scaling as for percolation. Then

$$\langle s^n \rangle_a = L^{d_a(n+1-\tau_a)} \mathcal{N}_a^{(n)}(L^\alpha \beta, L^{-\omega} u)$$

In two dimensions, mean-field avalanches corresponds to $\tau_a = \frac{5}{2}$ [4]. Such a scaling form is consistent with our measurement of the avalanche size distribution in fuse networks (Fig. 6), but only in the avalanche regieme; it does very poorly near percolation. In that limit, the distribution seems to collapse very well with only a rescaling of β.[6]

How do these scaling functions look for a system at critical stress (broken to within one avalanche of fracture)? Moments of the cluster size distribution should approach a constant as $\beta \to 0$ and vanish for $\beta \to \infty$, while moments of the avalanche size distribution should vanish for both $\beta \to 0$ and $\beta \to \infty$, but

[6] Until very small β, when the threshold distribution regularly samples numbers too small for every `long doubles` to hold.

take on finite values for an intermediate region. After expanding in $L^{-\omega}u$ to first order, we take

$$\langle s^n \rangle_x = L^{-d_x(n+1-\tau_x)} \left[\mathcal{J}_x^{(n)}(L^\alpha \beta) + L^{-\omega} \mathcal{K}_x^{(n)}(L^\alpha \beta) \right]$$

with

$$\mathcal{J}_a^{(n)}(X) = e^{-y_a^2(X)} \sum_{i=0}^{\infty} A_{ai}^{(n)} H_i\big(y_a(X)\big)$$

$$\mathcal{K}_a^{(n)}(X) = e^{-y_a^2(X)} \sum_{i=0}^{\infty} B_{ai}^{(n)} H_i\big(y_a(X)\big)$$

where $y_a(X) = \mu_a(\log X - \gamma_a)$ and

$$\mathcal{J}_c^{(n)}(X) = a^{(n)} \operatorname{erfc}\big(y_c(X)\big) + e^{-y_c^2(X)} \sum_{i=0}^{\infty} A_{ci}^{(n)} H_i\big(y_c(X)\big)$$

$$\mathcal{K}_c^{(n)}(X) = b^{(n)} \operatorname{erfc}\big(y_c(X)\big) + e^{-y_c^2(X)} \sum_{i=0}^{\infty} B_{ci}^{(n)} H_i\big(y_c(X)\big)$$

where $y_c(X) = \mu_c(\log X - \gamma_c)$. Fits to these functional forms are what led to the values of α and d_a used in the plots above.

This theory is already known [7]. We would like to extend it to describe the *process zone* of a crack, which is the microcracked region around its tip. Our principle idea is simple: take known scaling for finite-size effects in fracture, and in those laws replace L with x, the spatial distance from the crack tip. Since L and x have the same scaling dimension, they should contribute to scaling variables with the same exponent. However, for the quantities we want to study, this is not so simple. Specifically, we are interested in the stress, damage, and elastic modulus in the vicinity of a crack, which corresponds to current, damage, and conductivity in our models. Developing scaling theories for all three of these quantities has proved challenging, but we have ideas for how to do so.

The cases of damage and conductivity seems like it should be straightforward. After all, we already have a theory for the scaling of the percolation threshold p above! As before, the nth moment of the distribution of percolation thresholds in terms of $z = p - p_c$ is

$$\begin{aligned}
\langle z^n \rangle_p &= \int_{-p_c}^{1-p_c} z^n P(z + p_c \,|\, L, \beta)\, \mathrm{d}z = L^{1/\nu} \int_{-p_c}^{1-p_c} z^n \mathcal{P}(L^{1/\nu} z, L^\alpha \beta)\, \mathrm{d}z \\
&= L^{-n/\nu} \int_{-L^{1/\nu} p_c}^{L^{1/\nu}(1-p_c)} x^n \mathcal{P}(x, L^\alpha \beta)\, \mathrm{d}x \simeq L^{-n/\nu} \int_{-\infty}^{\infty} x^n \mathcal{P}(x, L^\alpha \beta) \mathrm{d}x \\
&= L^{-n/\nu} \mathcal{P}^{(n)}(L^\alpha \beta)
\end{aligned}$$

Note that we again assumed $\mathcal{P}(x, L^\alpha \beta)$ is highly peaked around $x = 0$, and that this peak is far from the boundaries of integration. If these assumptions do

not hold (as they do not for large β, when p_c becomes very small), we should expect other corrections. This means that moments of the percolation threshold distribution will have the scaling form

$$\langle p^n \rangle_p = \langle (z + p_c)^n \rangle_p = \sum_{i=1}^{n} \binom{n}{i} p_c^{n-i} L^{-i/\nu} \mathcal{P}^{(n)}(L^\alpha \beta)$$

While this is complex and does not admit a simple analysis, the moments with mean subtracted do,

$$\langle (p - \langle p \rangle)^n \rangle = L^{-n/\nu} \tilde{\mathcal{P}}^{(n)}(L^\alpha \beta)$$

where $\tilde{\mathcal{P}}^{(n)}$ is a new function consisting of a polynomial of $\mathcal{P}^{(i)}$ for $i = 1, \ldots, n$. Similarly, the conductivity scales with p like

$$P(\sigma \mid p) = \sigma^{-1} g\big(\sigma^{-1/t}(p - p_c)\big)$$

and so scales with L and β like

$$\begin{aligned} P(\sigma \mid L, \beta) &= \sigma^{-1} L^{1/\nu} \int g(\sigma^{-1/t}(p - p_c)) \mathcal{P}(L^{1/\nu}(p - p_c), L^\alpha \beta) \, \mathrm{d}p \\ &\simeq \sigma^{-1} \int g(L^{-1/\nu} \sigma^{-1/t} y) \mathcal{P}(y, L^\alpha \beta) \, \mathrm{d}y \\ &= \sigma^{-1} \mathcal{S}(L^{t/\nu} \sigma, L^\alpha \beta) \end{aligned}$$

This means that moments of the conductivity are given by

$$\begin{aligned} \langle \sigma^n \rangle &= \int \sigma^{n-1} \mathcal{S}(L^{t/\nu} \sigma, L^\alpha \beta) \, \mathrm{d}\sigma \\ &= L^{(1-n)t/\nu - t/\nu} \int x^{n-1} \mathcal{S}(x, L^\alpha \beta) \, \mathrm{d}x = L^{-nt/\nu} \mathcal{S}^{(n)}(L^\alpha \beta) \end{aligned}$$

These seem like very straightforward predictions with very straightforward scaling collapses. Fig. 7 shows the result in our numerics. The scaling predictions work very well for small β, in the vicinity of the fixed point, but fail in a way which increases with L in the avalanche regime! Clearly, there is another kind of scaling happening there. For instance, as β grows so does the expectation value of the conductivity, whose scaling theory in percolation no longer fully makes sense when the distribution's mean is nonzero. Perhaps by summing scaling contributions from two fixed points, the percolation one and the avalanche one, we can explain this data[7] and subsequently extend the analysis to the process zone.

Current is a whole other beast. Because critical current, as defined above, is proportional to the fuse threshold of the fuse that starts the last avalanche, the critical current scales roughly like $\langle p^{1/\beta} \rangle$. This vanishes as $\beta \to 0$ much

[7] And better explain our avalanche data!

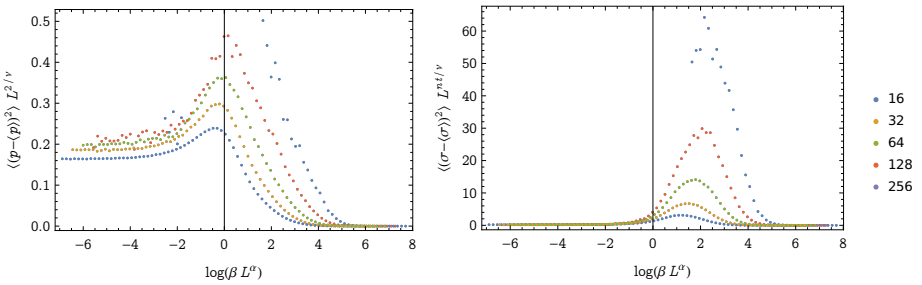

Figure 7: Plots of the second moment of the damage and conductivity size distributions normalized by scaling predictions.

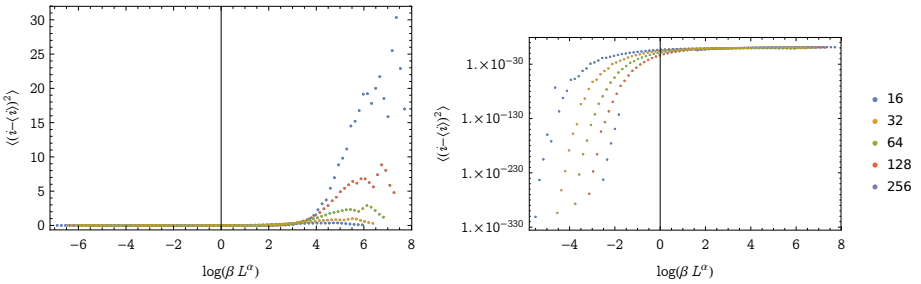

Figure 8: Plots of the second moment of the current distribution normalized by scaling predictions, both single-log and log–log.

faster than any power law and is hugely dependant on fluctuations in p (Fig. 8). Therefore, the current itself and anything that depends on the current (including the energy released in breaking) is sick, and we have not created a satisfactory scaling description for its behavior. Normalizing high moments by the second moment, however, does lead to reasonably good collapse (Fig. 9), but tells us relatively little about the way that stress itself, or its fluctuations, scale.

Finally, we have been thinking about ways of actually modelling the process zone in our numerics. One approach is to used notched networks, where fuses are cut along a line before the fracture is begun. One example of this can be seen in Fig. 10. These systems are very difficult to analyze for two reasons: they interact strongly with the boundaries, and do not represent the realistic presence of a crack, which in its growth has broken the regions surrounding it. Notice in the figure that regions along the side of the notch are relatively undamaged compared to regions in front of the notch. We have tried to alleviate these problems by choice of novel boundary conditions: voltage boundaries on all four sides of the system whose value reflects the value of voltage due to a semi-infinite crack in an unbroken system at that position—effectively embedding our fuse network in a static homogeneous semi-infinite cracked system. These simulations to a better job, but still lack realistic crack growth.

Another more recent idea has been to take a broken system and analyse the

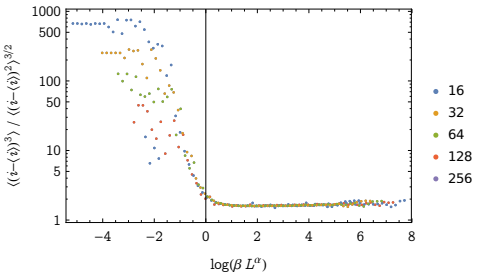

Figure 9: Plot of the third moment of the stress distribution normalized by the second moment.

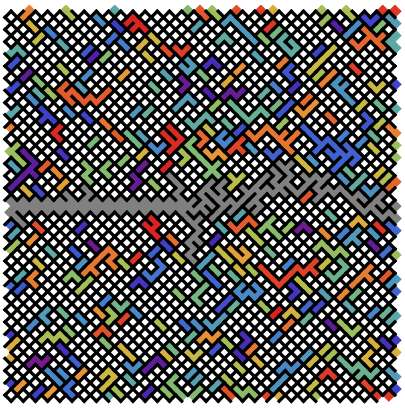

Figure 10: A broken network that began with a notch, with nontrivial clusters individually colored and the fracture surface highlighted.

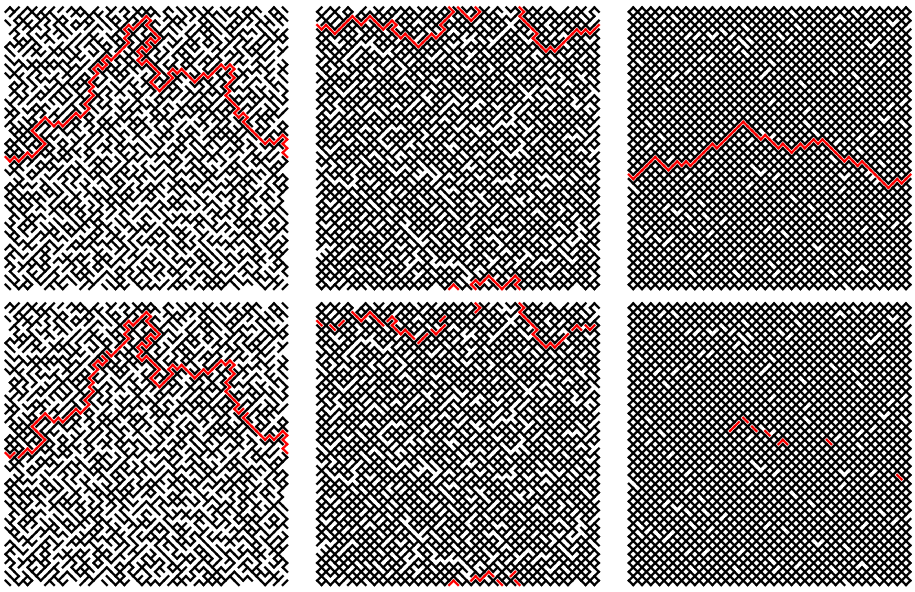

Figure 11: Top: Broken fuse networks for $\beta = 0.03$, 0.5, and 3 with the unique dual cycle highlighted. Bottom: The same networks and cycles, but at critical current.

precursors to the crack that has formed. In a broken fuse network, there is a unique cycle on the dual graph[8] which describes a minimal fracture surface (Fig. 11). Once identified, we can look at the vicinity of these regions in the critical network, or even before then, and measure damage and current as they relate to regions which are soon to become the spanning crack.

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[8] Actually, with torus boundaries there can be many cycles on the dual graph which correspond to loops along the strain direction, but there is only one cycle of homotopy class $(1, 0)$.

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