When is the average number of saddle points typical?

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Abstract – A common measure of a function's complexity is the count of its stationary points. For complicated functions, this count grows exponentially with the volume and dimension of their domain. In practice, the count is averaged over a class of functions (the annealed average), but the large numbers involved can produce averages biased by extremely rare samples. Typical counts are reliably found by taking the average of the logarithm (the quenched average), which is more difficult and not often done in practice. When most stationary points are uncorrelated with each other, quenched and annealed averages are equal. Equilibrium heuristics can guarantee when most of the lowest minima will be uncorrelated. We show that these equilibrium heuristics cannot be used to draw conclusions about other minima and saddles by producing examples among Gaussian-correlated functions on the hypersphere where the count of certain saddles and minima has different quenched and annealed averages, despite being guaranteed "safe" in the equilibrium setting. We determine conditions for the emergence of non-trivial correlations between saddles, and discuss the implications for the geometry of those functions and what out-of-equilibrium settings might be affected.

Random high-dimensional energies, cost functions, and interaction networks are important across disciplines: the energy landscape of glasses, the likelihood landscape of machine learning and inference, and the interactions between organisms in an ecosystem are just a few examples [1–4]. A traditional tool for making sense of their behavior is to analyze the statistics of points where their dynamics are stationary [5–8]. For energy or cost landscapes, these correspond to the minima, maxima, and saddles, while for ecosystems and other non-gradient dynamical systems these correspond to equilibria of the dynamics. When many stationary points are present, the system is considered complex.

Despite the importance of stationary point statistics for understanding complex behavior, they are often calculated using an uncontrolled approximation. Because their number is so large, it cannot be reliably averaged. The annealed approximation takes this average anyway, risking a systematic bias by rare and atypical samples. The annealed approximation is known to be exact for certain models and in certain circumstances, but it is used outside

those circumstances without much reflection [9–11]. In a few cases researchers have instead made the better-controlled quenched average, which averages the logarithm of the number of stationary points, and found deviations from the annealed approximation with important implications for behavior [12–18]. Generically, the annealed approximation to the complexity is wrong when a non-vanishing fraction of pairs of stationary points have non-trivial correlations in their mutual position.

A heuristic line of reasoning for the appropriateness of the annealed approximation is sometimes made when the approximation is correct for an equilibrium calculation on the same system. The argument goes like this: since the limit of zero temperature in an equilibrium calculation concentrates the Boltzmann measure onto the lowest set of minima, the equilibrium free energy in the limit to zero temperature will be governed by the same statistics as the count of that lowest set of minima. This argument is strictly valid only for the lowest minima, which at least in glassy problems are rarely relevant to dynamical behavior. What about the *rest* of the stationary points?

In this paper, we show that the behavior of the ground state, or *any* equilibrium behavior, does not govern

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whether stationary points will have a correct annealed average. In a prototypical family of models of random functions, we determine a condition for when annealed averages should fail and some stationary points will have non-trivial correlations in their mutual position. We produce examples of models whose equilibrium is guaranteed to never see such correlations between thermodynamic states, but where a population of saddle points is nevertheless correlated. The Supplementary Material `annealed.nb` recreates the data and figures of this paper.

We study the mixed spherical models, which are models of Gaussian-correlated random functions with isotropic statistics on the $(N-1)$-sphere. Each model consists of a class of functions $H: S^{N-1} \to \mathbb{R}$ defined by the covariance between the functions evaluated at two different points $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S^{N-1}$, which is a function of the scalar product (or overlap) between the two configurations:

$$\overline{H(\boldsymbol{\sigma}_1) H(\boldsymbol{\sigma}_2)} = \frac{1}{N} f\left(\frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{N} \right). \tag{1}$$

We will further take the distribution of H to be centered, *i.e.*, $\overline{H(\boldsymbol{\sigma})} = 0$ for all $\boldsymbol{\sigma} \in S^{N-1}$, which is equivalent to the absence of any deterministic term (or spike) in the function. Specifying the covariance function f uniquely specifies the model. The series coefficients of f need to be non-negative in order for f to be a well-defined covariance. The case where f is a homogeneous polynomial has been extensively studied, and corresponds to the pure spherical models of glass physics or the spiked tensor models of statistical inference [19]. Here our examples will be models with $f(q) = \frac{1}{2}(\lambda q^3 + (1-\lambda)q^s)$ for $\lambda \in (0,1)$, called $3+s$ models[1]. These are examples of *mixed* spherical models, which have been studied in the physics and statistics literature and host a zoo of complex orders and phase transitions [13,20–26].

There are several well-established results on the equilibrium of this model. First, if the function $\chi(q) = f''(q)^{-1/2}$ is convex then it is not possible for the equilibrium solution to have non-trivial correlations between states at any temperature [13,22][2]. This is a strong condition on the form of equilibrium order. Note that non-convex χ

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[1]Though the examples and discussion will focus on the $3+s$ models, most formulas (including the principal result in (12)) are valid for arbitrary covariance functions f under the condition that $f'(0) = 0$, *i.e.*, that there is no linear field in the problem. This condition is necessary to ensure that what we call "trivial" correlations are actually *zero* correlations: in the absence of a field, trivially correlated points on the sphere are orthogonal. This simplifies our formulas by setting the overlap q_0 between trivially correlated configurations to zero, which would otherwise be another order parameter, but reduces the scope of this study. The trivial overlap q_0 is also important in situations where a deterministic field (or spike) is present, as in [15], but deterministic fields are likewise not considered here.

[2]More specifically, convex χ cannot have an equilibrium order with more than 1RSB order among the configurations. In equilibrium, 1RSB corresponds to trivial correlations between thermodynamic states, but nontrivial correlations exist *within* a state at non-zero temperature. When temperature goes to zero, 1RSB in equilibrium reduces to replica symmetry among the lowest-lying states. Because in this paper we focus on symmetry breaking between stationary
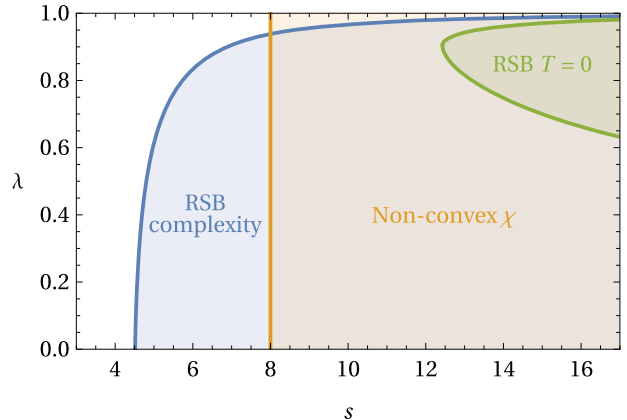

Fig. 1: A phase diagram of the boundaries we discuss in this paper for the $3+s$ model with $f = \frac{1}{2}(\lambda q^3 + (1-\lambda)q^s)$. The blue region shows models which have some stationary points with nontrivial correlated (RSB) structure, and is given by $G_f > 0$ where G_f is found in (12). The yellow region shows where $\chi(q) = f''(q)^{-1/2}$ is not convex and therefore non-trivial correlations between states are possible in equilibrium. The green region shows where non-trivial correlations exist at the ground state, adapted from [27]. We find that models where correlations between equilibrium states are forbidden can nonetheless harbor correlated stationary points.

does not imply that you *will* see non-trivial correlations between states at some temperature. In the $3+s$ models we consider here, models with $s > 8$ have non-convex χ and those with $s \le 8$ have convex χ independent of λ. Second, the characterization of the ground state has been made [13,20,23,27]. In the $3+s$ models we consider, for $s > 12.430\ldots$ non-trivial ground state configurations (more than 1RSB) appear in a range of λ. These bounds on equilibrium order are shown in fig. 1, along with our result for where the complexity has non-trivial correlations between some stationary points. As evidenced in that figure, correlations among saddles are possible well inside regions that forbid them among equilibrium states.

There are two important features which differentiate stationary points $\boldsymbol{\sigma}^*$ in the spherical models: their *energy density* $E = \frac{1}{N} H(\boldsymbol{\sigma}^*)$ and their *stability* $\mu = \frac{1}{N} \cdot \mathrm{Tr}\,\mathrm{Hess}\,H(\boldsymbol{\sigma}^*)$. The energy density gives the "height" in the landscape, while the stability governs the spectrum of the stationary point. In each spherical model, the spectrum of every stationary point is a Wigner semicircle of the same width $\mu_\mathrm{m} = \sqrt{4f''(1)}$, but shifted by constant. The stability μ sets this constant shift. When $\mu < \mu_\mathrm{m}$, the spectrum has support over zero and we have saddles with an extensive number of downward directions. When $\mu > \mu_\mathrm{m}$ the spectrum has support only over positive eigenvalues, and we have stable minima[3]. When $\mu = \mu_\mathrm{m}$, the spectrum has a pseudogap, and we have marginal minima.

points, we consider this form of RSB in equilibrium trivial because it does not imply any non-trivial correlations between states.

[3]Saddle points with a subextensive number of downward directions also exist via large deviations of some number of eigenvalues from the average spectrum.

$$\mathcal{D}(\mu) = \begin{cases} \dfrac{1}{2} + \log\left(\dfrac{1}{2}\mu_{\mathrm{m}}\right) + \dfrac{\mu^2}{\mu_{\mathrm{m}}^2}, & \mu^2 \le \mu_{\mathrm{m}}^2, \\[2ex] \dfrac{1}{2} + \log\left(\dfrac{1}{2}\mu_{\mathrm{m}}\right) + \dfrac{\mu^2}{\mu_{\mathrm{m}}^2} - \left|\dfrac{\mu}{\mu_{\mathrm{m}}}\right|\sqrt{\left(\dfrac{\mu}{\mu_{\mathrm{m}}}\right)^2 - 1} - \log\left(\left|\dfrac{\mu}{\mu_{\mathrm{m}}}\right| - \sqrt{\left(\dfrac{\mu}{\mu_{\mathrm{m}}}\right)^2 - 1}\right), & \mu^2 > \mu_{\mathrm{m}}^2. \end{cases} \tag{4}$$

The number $\mathcal{N}(E,\mu)$ of stationary points with energy density E and stability μ is exponential in N. Their complexity $\Sigma(E,\mu)$ is defined by the average of the logarithm of their number: $\Sigma(E,\mu) = \frac{1}{N}\overline{\log\mathcal{N}(E,\mu)}$. More often the annealed complexity is calculated, where the average is taken before the logarithm: $\Sigma_{\mathrm{a}}(E,\mu) = \frac{1}{N}\log\overline{\mathcal{N}(E,\mu)}$. The annealed complexity has been computed for these models [24,28], and the quenched complexity has been computed for a couple examples which have non-trivial ground states [13,16]. The annealed complexity bounds the complexity from above. A positive complexity indicates the presence of an exponentially large number of stationary points of the indicated kind, while a negative one means it is vanishingly likely they will appear. The line of zero complexity is significant as the transition between many stationary points and none.

In these models, trivial correlations between stationary points correspond to zero overlap: almost all stationary points are orthogonal to each other. This corresponds to *replica symmetric* (RS) order. The emergence of nontrivial correlations, and the invalidity of the annealed approximation, occurs when some non-vanishing fraction of stationary point pairs have a non-zero overlap. This corresponds to some kind of *replica symmetry breaking* (RSB). Here we restrict ourselves to a 1RSB ansatz, which corresponds to two kinds of pairs of stationary point: a fraction x of pairs have the trivial zero overlap, and the remaining fraction $1-x$ have a nontrivial overlap q_1. In the annealed or replica-symmetric case, $x=1$ and all but a vanishing fraction of stationary points are uncorrelated with each other. Since other kinds of RSB order encompass 1RSB, we are guaranteed that $\Sigma \le \Sigma_{1\mathrm{RSB}} \le \Sigma_{\mathrm{a}}$. We will discuss later in what settings the 1RSB complexity is correct.

When the complexity is calculated using the Kac-Rice formula and a physicists' tool set, the problem is reduced to the evaluation of an integral by the saddle point method for large N [16]. The complexity is given by extremizing an effective action,

$$\Sigma_{1\mathrm{RSB}}(E,\mu) =$$
$$\lim_{n\to 0}\int \mathrm{d}q_1\, \mathrm{d}x\, \mathcal{S}_{1\mathrm{RSB}}(q_1,x\mid E,\mu)e^{nN\mathcal{S}_{1\mathrm{RSB}}(q_1,x\mid E,\mu)}$$
$$= \operatorname*{extremum}_{q_1,x}\mathcal{S}_{1\mathrm{RSB}}(q_1,x\mid E,\mu) \tag{2}$$

for the action $\mathcal{S}_{1\mathrm{RSB}}$ given by

$$\mathcal{S}_{1\mathrm{RSB}}(q_1,x\mid E,\mu) = \mathcal{D}(\mu) + \operatorname*{extremum}_{\hat\beta,r_{\mathrm{d}},r_1,d_{\mathrm{d}},d_1}\left\{\hat\beta E - r_{\mathrm{d}}\mu\right.$$
$$+ \frac{1}{2}\Big[\hat\beta^2[f(1) - \Delta x f(q_1)] + (2\hat\beta r_{\mathrm{d}} - d_{\mathrm{d}})f'(1)$$

$$- \Delta x(2\hat\beta r_1 - d_1)f'(q_1) + r_{\mathrm{d}}^2 f''(1) - \Delta x\, r_1^2 f''(q_1)$$
$$+ \frac{1}{x}\log((r_{\mathrm{d}} - \Delta x\, r_1)^2 + d_{\mathrm{d}}(1 - \Delta x\, q_1) - \Delta x\, d_1(1 - \Delta x q_1))$$
$$- \frac{\Delta x}{x}\log((r_{\mathrm{d}} - r_1)^2 + (d_{\mathrm{d}} - d_1)(1 - q_1))\Big]\Big\}, \tag{3}$$

where $\Delta x = 1 - x$ and

see eq. (4) above

The details of the derivation of these expressions can be found in [16]. The extremal problem in $\hat\beta$, r_{d}, r_1, d_{d}, and d_1 has a unique solution and can be found explicitly, but the resulting formula is unwieldy. The action can have multiple extrema, but the one for which the complexity is *smallest* gives the correct solution. There is always a solution for $x=1$ which is independent of q_1, corresponding to the replica symmetric case, and with $\Sigma_{\mathrm{a}}(E,\mu) = \mathcal{S}_{1\mathrm{RSB}}(E,\mu\mid q_1,1)$. The crux of this paper will be to determine when this solution is not the global one.

It is not accurate to say that a solution to the saddle point equations is "stable" or "unstable". The problem of solving the complexity in this way is not a variational problem, so there is nothing to be maximized or minimized, and in general even global solutions are not even local minima of the action. However, the stability of the action can still tell us something about the emergence of new solutions: when a new solution bifurcates from an existing one, the action will have a flat direction. Unfortunately this is difficult to search out, since one must know the parameters of the new solution, and q_1 is unconstrained and can take any value in the old solution.

There is one place where we can consistently search for a bifurcating solution to the saddle point equations: along the zero complexity line $\Sigma_{\mathrm{a}}(E,\mu) = 0$. Going along this line in the replica symmetric solution, the 1RSB complexity transitions at a critical point where $x = q_1 = 1$ [16]. Since all the parameters in the bifurcating solution are known at this point, we can search for it by looking for a flat direction. In the annealed solution for points describing saddles (with $\mu^2 \le \mu_{\mathrm{m}}^2$ and therefore the simpler form of (4)), this line is

$$\mu_0 = -\frac{2Ef'f''}{z_f} - \sqrt{\frac{2f''u_f}{z_f^2}\left(\log\frac{f''}{f'}z_f - E^2(f'' - f')\right)}, \tag{5}$$

where we have chosen the lower branch as a convention (see fig. 2) and where we define for brevity (here and elsewhere) the constants

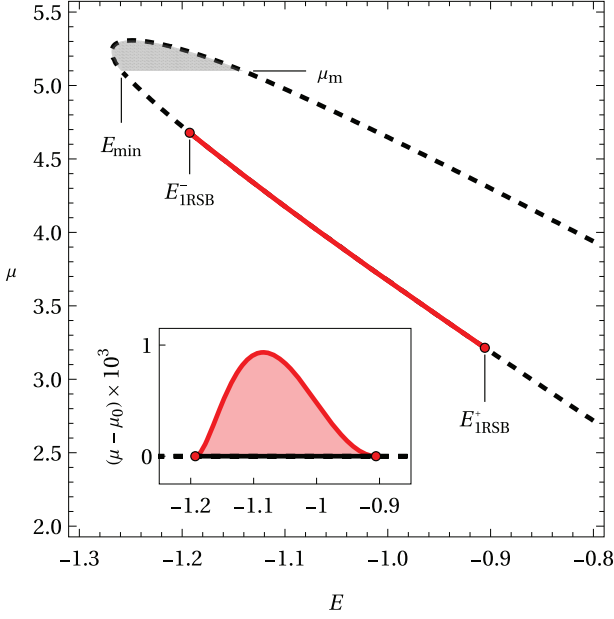$$u_f = f(f' + f'') - f'^2, \quad v_f = f'(f'' + f''') - f''^2,$$

Fig. 2: Stationary point statistics as a function of energy density E and stability μ for a model with $f(q) = \frac{1}{2}(\frac{1}{2}q^3 + \frac{1}{2}q^5)$. The dashed black line shows the line of zero annealed complexity and enclosed inside the annealed complexity is positive. The solid black line (only visible in the inset) gives the line of zero 1RSB complexity. The red region (blown up in the inset) shows where the annealed complexity gives the wrong count and a 1RSB complexity in necessary. The red points show where $\det M = 0$. The left point, which is only an upper bound on the transition, coincides with it in this case. The gray shaded region highlights the minima, which are stationary points with $\mu \geq \mu_{\mathrm{m}}$. E_{min} is marked on the plot as the lowest energy at which saddles of extensive index are found.

$$w_f = 2f''(f'' - f') + f'f''', \quad y_f = f'(f' - f) + f''f,$$
$$z_f = f(f'' - f') + f'^2. \tag{6}$$

When f and its derivatives appear without an argument, the implied argument is always 1, so, e.g., $f' \equiv f'(1)$. If f has at least two non-zero coefficients at second order or higher, all of these constants are positive. Though in figures we focus on the lower branch of saddles, another set of identical solutions always exists for $(E, \mu) \mapsto (-E, -\mu)$. We also define E_{min}, the minimum energy at which saddle points with an extensive number of downward directions are found, as the energy for which $\mu_0(E_{\mathrm{min}}) = \mu_{\mathrm{m}}$.

Let M be the matrix of double partial derivatives of the action with respect to q_1 and x. We evaluate M at the replica symmetric saddle point $x = 1$ with the additional constraint that $q_1 = 1$ and along the extremal complexity line (5). We determine when a zero eigenvalue appears, indicating the presence of a bifurcating 1RSB solution, by solving $0 = \det M$. We find

$$\det M = -\left(\frac{\partial^2 \mathcal{S}_{\mathrm{1RSB}}}{\partial q_1 \partial x}\bigg|_{\substack{x=1 \\ q_1=1}}\right)^2 \propto (ay^2 + bE^2 + 2cyE - d)^2, \tag{7}$$

where $y = -\frac{1}{2}z_f\mu - f'f''E$ is proportional to the square-root term in (5) and the constants a, b, c, and d are defined by

$$a = \frac{w_f(3y_f^2 - 4ff'f''(f' - f)) - 6y_f^2(f'' - f')f''}{(u_f z_f f'')^2 f'}$$
$$b = \frac{f'w_f}{z_f^2}, \qquad c = \frac{w_f}{f''z_f^2}, \qquad d = \frac{w_f}{f'f''}. \tag{8}$$

Changing variables from μ to y is convenient because the branch of (5) is chosen by the sign of y (the lower-energy branch we are interested in corresponds to $y > 0$). The relationship between y and E on the extremal line is $g = 2hy^2 + eE^2$, where the constants e, g, and h are given by

$$e = f'' - f', \qquad g = z_f \log\frac{f''}{f'}, \qquad h = \frac{1}{f''u_f}. \tag{9}$$

The solutions for $\det M = 0$ can be calculated explicitly and correspond to energies that satisfy

$$E_{\mathrm{1RSB}}^{\pm} = \frac{\operatorname{sign}(bg - de)\left(-cg \pm \sqrt{\Delta_f}\right)}{\sqrt{2c^2eg + (2bh - ae)(bg - de) \mp 2ce\sqrt{\Delta_f}}}, \tag{10}$$

where the discriminant Δ_f is given by

$$\Delta_f = c^2g^2 + (2dh - ag)(bg - de). \tag{11}$$

This predicts two points where a 1RSB solution can bifurcate from the annealed one. The remainder of the transition line can be found by solving the extremal problem for the action very close to one of these solutions, and then taking small steps in the parameters E and μ until it terminates. In many cases considered here, the line of transitions in the complexity that begins at E_{1RSB}^+, the higher energy point, ends exactly at E_{1RSB}^-, the lower energy point, so that these two points give the precise range of energies at which RSB saddles are found. An example that conforms with this picture for a $3 + 5$ mixed model is shown in fig. 2. In that figure, the range of μ with 1RSB ordering at any fixed E is extremely small. With increasing s the range also increases (see the example of the $3 + 16$ model in [16]), but we do not have any intuition for why this happens.

The discriminant Δ_f inside the square root of (10) is proportional to

$$G_f = f'\log\frac{f''}{f'}[3y_f(f'' - f')f''' - 2(f' - 2f)f''w_f]$$
$$- 2(f'' - f')u_fw_f - 2\log^2\frac{f''}{f'}f'^2f''v_f. \tag{12}$$

If $G_f > 0$, then the bifurcating solutions exist, and there are some saddles whose complexity is corrected by a 1RSB solution. Therefore, $G_f > 0$ is a sufficient condition to see at least 1RSB in the complexity. If $G_f < 0$, then there is nowhere along the extremal line where saddles can be described by such a complexity, but this does not definitively rule out RSB: the model may be unstable to different
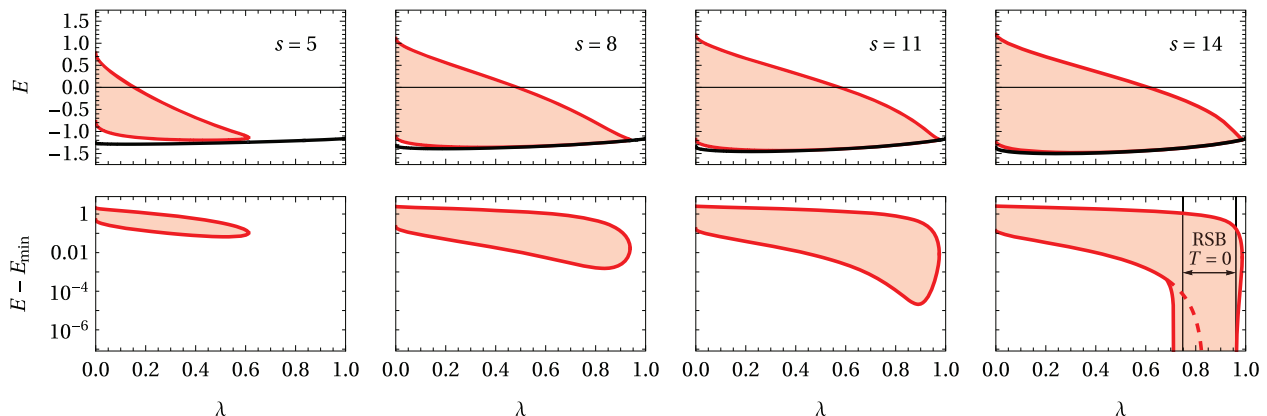
Fig. 3: The range of energies where RSB saddles are found for the $3+s$ model with varying s and λ. In the top row the black line shows $E_{\min}$, the minimum energy where saddles are found, and in the bottom row this energy is subtracted away to emphasize when the RSB region crosses into minima. For most s, both the top and bottom lines are given by $E_{1\text{RSB}}^{\pm}$, but for $s = 14$ there is a portion where the low-energy boundary has $q_1 < 1$. In that plot, the continuation of the $E_{1\text{RSB}}^{-}$ line is shown dashed. Also marked is the range of λ for which the ground state minima are characterized by non-trivial RSB.

RSB orders, or its phase boundary may simply not have a critical point on the extremal line. We discuss the former possibility later in the paper. The range of $3 + s$ models where G_f is positive is shown in fig. 1.

Figure 3 shows the range of energies where non-trivial correlations are found between stationary points in several $3 + s$ models as λ is varied. For models with smaller s, such correlations are found only among saddles, with the boundary never dipping beneath the minimum energy of saddles $E_{\min}$. Also, these models have a transition boundary that smoothly connects $E_{1\text{RSB}}^{+}$ and $E_{1\text{RSB}}^{-}$, so $E_{1\text{RSB}}^{-}$ corresponds to the lower bound of RSB complexity. For large enough s, the range passes into minima, which is expected as these models have non-trivial complexity of their ground states. Interestingly, this appears to happen at precisely the value of s for which non-trivial ground state configurations appear, $s = 12.430\ldots$. This also seems to correspond with the decoupling of the RSB solutions connected to $E_{1\text{RSB}}^{+}$ and $E_{1\text{RSB}}^{-}$, with the two phase boundaries no longer corresponding, as in fig. 4. In these cases, $E_{1\text{RSB}}^{-}$ sometimes gives the lower bound, but sometimes it is given by the termination of the phase boundary extended from $E_{1\text{RSB}}^{+}$.

There are implications for the emergence of RSB in equilibrium. Consider a specific H with

$$H(\boldsymbol{\sigma}) = \frac{\sqrt{\lambda}}{p!} \sum_{i_1 \cdots i_p} J_{i_1 \cdots i_p}^{(p)} \sigma_{i_1} \cdots \sigma_{i_p}$$
$$+ \frac{\sqrt{1-\lambda}}{s!} \sum_{i_1 \cdots i_s} J_{i_1 \cdots i_s}^{(s)} \sigma_{i_1} \cdots \sigma_{i_s}, \qquad (13)$$

where the interaction tensors J are drawn from zero-mean normal distributions with $\overline{(J^{(p)})^2} = p!/2N^{p-1}$ and likewise for $J^{(s)}$. Functions H defined this way have the covariance property (1) with $f(q) = \frac{1}{2}\big(\lambda q^p + (1-\lambda)q^s\big)$. With the J's drawn in this way and fixed for $p = 3$ and $s = 14$, we can
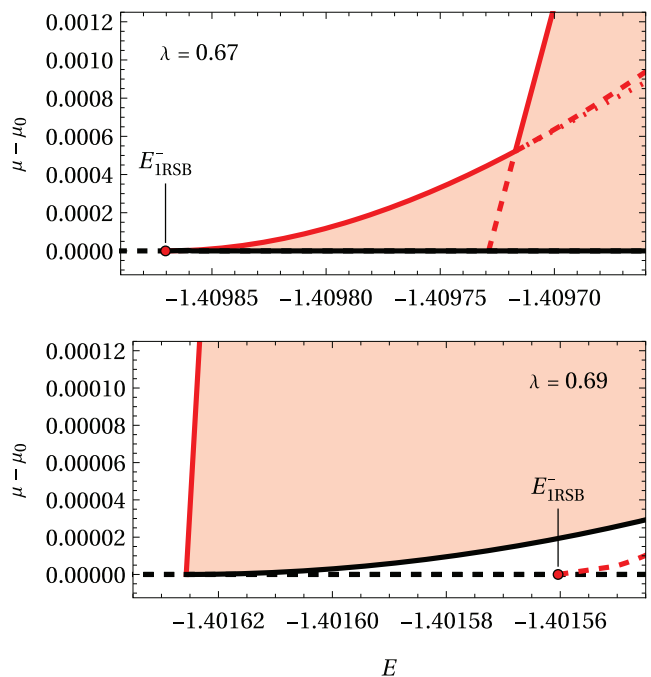

Fig. 4: Examples of $3+14$ models where $E_{1\text{RSB}}^{-}$ (top) is the lower bound and (bottom) is not the lower bound of energies with RSB saddles. In both plots the red dot shows $E_{1\text{RSB}}^{-}$, while the solid red lines show the transition boundary between RS and 1RSB complexities. The dashed black line shows the RS zero complexity line, while the solid black line shows the 1RSB zero complexity line. The dashed red lines show the spinodals of the 1RSB phases. The dotted red line shows a discontinuous phase transition between different 1RSB phases. Top: $\lambda = 0.67$. The transition line that begins at $E_{1\text{RSB}}^{+}$ does not intersect $E_{1\text{RSB}}^{-}$ but terminates at a higher energy. $E_{1\text{RSB}}^{-}$ is a lower bound on the energy of RSB saddles. There are two competing 1RSB phases among saddles. Bottom: $\lambda = 0.69$. The transition line that begins at $E_{1\text{RSB}}^{+}$ terminates at a lower energy than $E_{1\text{RSB}}^{-}$, and therefore its terminus defines the lower bound.

vary λ, and according to fig. 1 we should see a transition in the type of order at the ground state. What causes the change? Our analysis indicates that stationary points with the required order *already exist in the landscape* as unstable saddles for small λ, then eventually stabilize into metastable minima and finally become the lowest lying states. This is different from the picture of existing uncorrelated low-lying states splitting apart into correlated clusters. Where uncorrelated stationary points do appear to split apart, when λ is decreased from large values, is among saddles, not minima.

A similar analysis can be made for other mixed models, like the $2 + s$, which should see complexities with other forms of RSB. For instance, in [16] we show that the complexity transitions from RS to full RSB (FRSB) along the line $y = mx + b$

$$\mu = -\frac{(f' + f''(0))u_f}{(2f - f')f'f''(0)^{1/2}} - \frac{f'' - f'}{f' - 2f}E, \qquad (14)$$

which can only be realized when $f''(0) \neq 0$, as in the $2 + s$ models. For $s > 2$, this transition line *always* intersects the extremal line (5), and so RSB complexity will always be found among some population of stationary points. However, it is likely that for much of the parameter space the so-called one-full RSB (1FRSB), rather than FRSB, is the correct solution, as it likely is for large s and certain λ in the $3 + s$ models studied here. Further work to find the conditions for transitions of the complexity to 1FRSB and 2FRSB is necessary. For values of s where there is trivial RSB in the ground state, we expect that the 1RSB complexity is correct.

What are the implications for dynamics? We find that non-trivial correlations tend to exist among saddle points with the largest or smallest possible index at a given energy density, which are quite atypical in the landscape. However, these strangely correlated saddle points must descend to uncorrelated minima, which raises questions about whether structure on the boundary of a basin of attraction is influential to the dynamics that descends into that basin. These saddles might act as early-time separatrices for descent trajectories of certain algorithms. With open problems in even the gradient decent dynamics on these models (itself attracted to an atypical subset of marginal minima), it remains to be seen whether such structures could be influential [28–30]. This structure among saddles cannot be the only influence, since it seems that the $3 + 4$ model is "safe" from non-trivial RSB among saddles.

We have determined the conditions under which the complexity of the mixed $3 + s$ spherical models has different quenched and annealed averages, as the result of non-trivial correlations between stationary points. We saw that these conditions can arise among certain populations of saddle points even when the model is guaranteed to lack such correlations between equilibrium states, and exist for saddle points at a wide range of energies. This suggests that studies making complexity calculations cannot reliably use equilibrium behavior to defend the annealed approximation. Our result has direct implications for the geometry of these landscapes, and perhaps could be influential to certain out-of-equilibrium dynamics.

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